UNIT - I **REAL ANALYSIS**

Definition:- Countable set :-

A set A is said to be countably infinite if A is a equivalent to the set of natural numbers N.

A is said to be countable if it is finite (or) countably infinite.

Note :

Let A be a countably infinite set. Then there is a bijection $f: N \to A$

Let $f(1) = a_1$, $f(2) = a_2 \dots f(n) = a_n$

Then $A = \{a_1, a_2 \dots a_n \dots\}$

Thus all the elements of A can be eabelled by using the elements of n.

Example:1

$\{2, 4, 6 \dots 2n \dots\}$ is a countable set.

Example: 2

Z is countable

Let $A = \{1/2, 2/3, 3/4 \dots\}$ the function $f: N \to A$ defined by $f(n) = \frac{n}{n+1}$ is a bijection.

Let $N = \{1, 2, 3, \ldots\}$

$A = \{1/2, 2/3, 3/4 \ldots\}$

A function $f(n) = n/(n+1)$

Let $n = 1$

$f(1) = \frac{1}{1+1} = 1/2$

$n = 2$

$f(2) = 2/3$

$f(3) = 3/4$

$A = \{1/2, 2/3, 3/4, \ldots\}$

Hence A is countable

Theorem: 1.1

The subset of countable is countable

Proof:

Let A be a countable set and Let $B \subseteq A$

If A (or) B is finite

Then obiviously B is uncountable

Hence Let A and B both infinite

since A is countably finite. we can write

$A = \{a_1 . a_2 \dots a_n \dots\}$

Let

$a_{n_1} \in B$ (a_{n_1} be the finite element in A such that $a_n \in B$)

Let a_{n_2} be the finite element in a A. Which follows a_n such that $a_{n_2} \in B$

proceeding like this we get

$B = \{a_{n_1}, a_{n_2} \dots\}$

Thus all the elements of B can be Labelled by using the elements of N.

Hence B is countable.

Theorem : 1.2

prove that Q^+ is countable

proof

Take all positive rational numbers where numerator and dinominator odd up to 2

we have only one number namely $1/1$

Next we take all positive rational number whose numerator and dinominator add up to 3.

We have $1/2$ and $2/1$

Next we take all positive rational numbers whose neumerator and dinominator add up to 4.

We have $2/2$, $1/3$ and $3/1$

proceeding like this we can list all the positive rational numbers together from the begining omitting those which are already listed

Thus we obtain the set

$\{1/1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4\}$

This set contains every positive rational numbers each occuring exactly once.

Thus Q^+ is countable

Theorem: 1.3

prove that Q is countable

proof:

First to prove that Q' is countable

Next to prove that Q is countable

let $Q' = \{r_1, r_2 \dots\}$

Then $Q = \{0, \pm r_1, r_2 \dots\}$ define a function $f : N \to Q$ by $f(1) = 0$

$f(2n) = r_n$

$f(2n+1) = -r_n$

clearly f is bijective function

Q is equivalent to N.

Q is countably infinite

Hence Q is countable.

Theorem: 1.4

prove that $N \times N$ is countable

proof:-

$$\boxed{N \times N = \{(a,b) / a,b \in N\}}$$

Take all order pairs $(a,b)/a+b = 2$

There exists only one pair

$(1,1) = 2$

Next we take all order pair

$(a,b)/a+b = 3$

There exists (2,1) and (1,2) = 3

Next we take all order pair

(a,b) / a+b = 4

There exists (2,2) (1,3) (3,1) = 4

proceeding like this we get all order pairs

$N \times N = \{(1,1) (1,2) (2,1) (2,2) (1,3) (3,1)\}$

This set contains every order pairs on $N \times N$ exactly once.

Hence $N \times N$ is countable.

Theorem : 1.5

If A and B are countable sets then $A \times B$ is also countable.

proof:-

first we prove that $N \times N$ is countable

Let A and B are countable set

To prove:

$A \times B$ is countable

Assume that:

$A = \{a_1, a_2 \ldots a_n \ldots\}$ and

$B = \{b_1, b_2 \ldots b_n \ldots\}$

Now defined $f: N \times N \to A \times B$ by

$f(i, j) = (a_i, b_j)$

We claim that :

F is objective function

suppose $x, y \in N \times N$

$x = (p, q)$ and $y = (u, v)$

$\Rightarrow f(x) = f(y)$

$\Rightarrow f(pq) = f(u, v)$

$\Rightarrow (a_p, b_q) = (a_u, b_v)$

$\Rightarrow a_p = a_u$ and $b_q = b_v$

$\Rightarrow p = u$ and $q = v$

$\Rightarrow (p, q) = (u, v)$

$\Rightarrow x = y$

f is one to one

Let $(a_m, b_n) \in A \times B$

There exists an $m, n \in N \times N$

such that

$f(m, n) = (a_m, b_n)$

$f(x) = y$

f is onto

clearly f is bijective function.

Hence $A \times B$ is equivalent to a countable set of $N \times N$

(ie) also $N \times N$ is countable

Therefore $A \times B$ is countable.

Theorem : 1.6

Let A, B of countably infinite set and f be a mapping of A onto a set B.

Then B is countable.

proof :

Let A be a countabley infinite set and $f : A \to B$ be an onto map.

Let $b \in B$

since f is onto

There exists atleast one pre image for b.

choose one element $a \in A$ such that

$$f(a) = b$$

Now defining $g : B \to A$ by $g(b) = a$

clearly $f(a) = g(b)$

clearly ψ is one to one

Therefore B is equivalent to the subset of the countable set A.

Therefore B is countable

[by theorem 1.1]

Theorem: 1.7

countable union of the countable set is countable (or) If $A_1, A_2, \ldots A_n$ are countable sets then $\bigcup_{n=1}^{\infty} A_n$ is countable.

proof:

Let A_i is countably infinite set.

Let us

$S = \{A_1, A_2 \ldots A_n \ldots\}$ is a countably infinite family of set.

case (i)

Let $A_1 = \{a_{11}, a_{12}, \ldots a_{1n} \ldots\}$

$A_2 = \{a_{21}, a_{22} \quad a_{2n} \ldots\}$

$\vdots$

$A_n = \{a_{n1}, a_{n2} \ldots a_{nn} \ldots\}$

Defined $f: N \times N \to$ union of A_n by

$f(i,j) = (a_{ij})$

Clearly f is onto.

W.K.T $N \times N$ is countably infinite

Also WKT a map f: a countably infinite set A into a set B is onto

f is onto

Let us $f: N \times N \to$ union of An is the bijective function.

Hence union of An is countable.

Case: (ii)

Let ~~Each~~ Ai be a countable sets for each i.

choose ~~a~~ set Bi such that Bi is the countably infinite sets and $Ai \subseteq Bi$

Then $\cup Ai \subseteq \cup Bi$

$\cup Bi$ is countable (by case i)

Therefore $\cup Ai$ is countable

[since by theorem 1.1]

problems:

Any countably infinite set is equivalent to be a proper subset of itself

soln: let A be a countably infinite set

let us

$$A = \{a_1, a_2, \ldots a_n, \ldots\}$$

$$\text{let } B = \{a_2, a_3 \ldots a_{n+1} \ldots\}$$

clearly

B is the proper subset of A define a map $f: A \to B$ by

$$f(a_n) = f(a_{n+1})$$

Clearly f is a bijection

Hence A is equivalent to B

2 Any infinite sets contains a countably infinite subset.

soln:-

let A be an infinite set

choose one elements $a \in A$

Since A is infinite set

we can choose another elements $a_2 \in A - \{a\}$

Now, suppose we have choose an. $a_1, a_2 \ldots a_n$ from A.

Since A is infinite set.

$A = \{a_1, a_2, \ldots a_n \ldots\}$ is also an infinite set.

We can choose a_{n+1} from $A = \{a_1, a_2 \ldots a_n \ldots\}$

Now $B = \{a_1, a_2 \ldots a_n, a_{n+1} \ldots\}$ is a countably infinite of subset of A.

5. Any infinite set is equivalent to a proper subset of itself

Soln:-

Let a be an infinite set WKT

Any infinite set contains countably infinite subset.

The set A contains a countably infinite subsets

Let $B = \{a_1, a_2 \ldots a_n \ldots\}$

$C = B - \{a_1\}$

ie) $C = \{a_2, a_3, \ldots a_n \ldots\}$

Clearly,

C is the proper subsets of A

To prove that:

A is equivalent to C

consider the function $f: A \to C$ defined by

$f(a_n) = a_{n+1} \ \forall \ a_n \in B$

clearly

f is bijective function.

A is equivalent to C.

Uncountable set:

A set which is not countable is called uncountable set.

Theorem: 1.8

$(0, 1]$ is uncountable.

Soln:-

Every real number any internal $(0, 1]$ can be written uniquelly as not terminating decimal

$0, a_1, a_2 \dots a_n \dots$

where $0 < a_i < 9$ for each subject to the following restriction, that any terminating dicimal as

$0, a_1, a_2 \dots (a_{n-1}) 99$

for example

$0.54 = 0.53999\ldots$

$0.1 = 0.0999\ldots$

To prove $[0,1]$ is uncountable

Suppose $[0,1]$ is countable

Then the elements of $[0,1]$ can be listed $\{x_1, x_2 \ldots x_n \ldots\}$

where

$x_1 = 0, a_{11}, a_{12} \ldots a_{1n} \ldots$

$x_2 = 0, a_{21}, a_{22} \ldots a_{2n} \ldots$

$\vdots$

$x_n = 0, a_{n1}, a_{n2} \ldots a_{nn} \ldots$

$\vdots$

For each positive integer n choose an integer b_n such that $0 < b_n < 999$

$b_n \neq 0$ and $b_n \neq a_{nn}$

Let $y = 0, b_1, b_2 \ldots$

clearly $y \in [0,1]$

Now y is different from each x_i atleast in the ith place

$\therefore y \neq x_i$ for each i

which is contradiction.

Hence $[0,1]$ is uncountable.

Corollary: 1

Any subset A of are which contains $[0.1]$ is uncountable.

proof

$A \subseteq R$

$[0,1] \subseteq A$

suppose A is countable.

To prove:-

A is countable.

$[0.1] \subseteq A$ [by theorem 1.1]

A subset of a $[0,1]$ is countable

which is contradiction.

$[0,1]$ is uncountable.

A is uncountable

corollary: 2

R is uncountable.

proof:-

To prove R is uncountable

suppose R is countable.

$[0,1] \subseteq R$

A subset of a countable set is countable.

$\therefore [0,1]$ is countable

which is contradiction.

Hence R is uncountable.

Corollary: 3.

The set S of irrational number is uncountable

proof:

Let S be a irrational

W.K.T

Q is the set of all rational numbers which is countable

To prove that:

S is uncountable

Suppose S is countable

$S \cup Q$ is also countable.

$R = S \cup Q$ is also countable

$\therefore$ which is contradiction.

Because R is uncountable.

Hence the set of all irrational number is uncountable.

State and prove HOLDER'S:-

INEQUALITIES:-

Statement:-

If $p > 1$ and q is such that $1/p + 1/q = 1$ then $\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \cdot \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}$ where $a_1, a_2, \ldots a_n$ and $b_1, b_2 \ldots b_n$ are real numbers

proof:-

If $p > 1$ and $1/p + 1/q = 1$

First we have to prove that

$x^{1/p} \cdot y^{1/q} \leq x/p + y/p$ where $x, y \geq 0$

if $x = 0$ $y = 0$ then the inequalities

Thus clearly the prove. $x \geq 0$ and $y \geq 0$

consider a function.

$f(t) = t^{\lambda} - \lambda t + \lambda - 1 = 0$

where $\lambda = 1/p$ $t \geq 0$

$$f'(t) = \lambda t^{\lambda - 1} - \lambda$$

$$f'(t) = \lambda (t^{\lambda - 1} - 1)$$

If $t = 1$ sub the function $f(t)$ and $f'(t)$

$$f(1) = 1 - \lambda + \lambda - 1$$

$$f(1) = 0$$

$f'(1) = \lambda(1-1) = 0$

$f'(1) = 0$

Also $f'(t) = 0$ where $0 < t < 1$

$f'(t) < 0$

where $t < 0$

Now $f'(t) =$ for every $t > 0$

put $t = x/y$

$f(t) = f(x/y) \leq 0$

$\Rightarrow (x/y)^{\cancel{1/p}} - \lambda(x/y) + \lambda - 1 = 0 \quad (\lambda = 1/1)$

$\Rightarrow (x/y)^{1/p} - 1/p\,(x/y) + 1/p - 1 = 0$

Multiplying in y on both sides

$\Rightarrow (x/y)^{1/p} y - 1/p\,(x/y)\,y + 1/p^{y} - 1 \cdot y \leq 0$

$\Rightarrow \left(\frac{x}{y}\right)^{1/p} \cdot y - 1/p\,(x) + y/p - y \leq 0$

$\Rightarrow \frac{x^{1/p} \cdot y}{y^{1/p}} - \frac{x}{p} + \frac{y}{p} - y \leq 0$

$\Rightarrow x^{1/p} \cdot y^{-1/p} \cdot y - x/p + y\,(1/p - 1) \leq 0$

$\Rightarrow x^{1/p} \cdot y^{1-1/p} - x/p + y\,(1/p - 1) \leq 0$

$\Rightarrow x^{1/p} \cdot y^{-1/p} - x/p + y\,(1/q) \leq 0$

$\Rightarrow \boxed{x_j^{1/p} \cdot y_j^{1/q} \leq x_j/p + y_j/q} \to (1)$

Consider $j = 1, 2, 3, \ldots n$

$$xj = \frac{|(ai)|^p}{\sum_{i=1}^{n} |(ai)|^p} \; ; \; yj = \frac{|bj|^q}{\sum_{i=1}^{n} |bi|^q}$$

$$(xj)^{1/p} = \frac{(|aj|^p)^{1/p}}{\left(\sum_{i=1}^{n} |ai|^p\right)^{1/p}}$$

$$(yj)^{1/p} = \frac{(|bj|^q)^{1/p}}{\left(\sum_{i=1}^{n} |bi|^q\right)^{1/p}}$$

This are the values sub in equ ①

$$\frac{(|aj|^p)^{1/p}}{\left(\sum_{i=1}^{n} |ai|^p\right)^{1/p}} \cdot \frac{(|bi|^q)^{1/p}}{\left(\sum_{i=1}^{n} |bi|^q\right)^{1/p}} \leq$$

$$(1/p)\left(\frac{|aj|^p}{\sum_{i=1}^{n} |ai|^p}\right) + 1/q \left(\frac{|bj|^q}{\sum_{i=1}^{n} |bi|^q}\right)$$

adding this inequalities:-

$$\frac{\sum_{j=1}^{n} |ajbj|}{\left(\sum_{i=1}^{n} |ai|^p\right)^{1/p} \cdot \left(\sum_{i=1}^{n} |bi|^q\right)^{1/q}}$$

$$\leq 1/p \left[\frac{\sum_{j=1}^{n} |aj|^p}{\sum_{i=1}^{n} |ai|^p}\right] + 1/q \left[\frac{\sum_{j=1}^{n} |bj|^q}{\sum_{i=1}^{n} |bj|^q}\right]$$

$$\frac{\sum_{j=1}^{n} |a_j| \, |b_j|}{\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}} \leq 1/p + 1/q$$

(where $i, j = 1, 2, 3 \ldots n$)

$$\frac{\sum_{j=1}^{n} |a_j| \, |b_j|}{\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_j|^q\right)^{1/q}} \leq 1$$

$$\sum_{j=1}^{n} |a_j| \, |b_j| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \cdot \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}$$

$$\sum_{i=1}^{n} |a_i b_j| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}$$

State and prove minkowski Inequality:

Statement :-

If $p \geq 1$
$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}$$

where $a_1, a_2 \ldots a_n$ and $b_1, b_2 \ldots b_n$ are real numbers.

proof:

Thus inequality is trivial when $p=1$

Let $p>1$ $|\cancel{a_i}+\cancel{b_i}|$

$$|a_i+b_i| \le |a_i|+|b_i|$$

$$\left(\sum_{i=1}^{n}|a_i+b_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n}\left((|a_i|+|b_i|)^p\right)\right)^{1/p} \rightarrow (1)$$

consider.

$$\sum_{i=1}^{n}\left((|a_i|+|b_i|)\right)^p = \sum_{i=1}^{n}(|a_i|+|b_i|)^{p-1}$$

$$=\sum_{i=1}^{n}|a_i|(|a_i|+|b_i|)^p + \sum_{n=1}^{n}|b_i|(|a_i|+|b_i|)$$

$$\sum_{i=1}^{n}|a_i| \sum_{j=1}^{n}(|a_i|+|b_j|)^{p-1} + \sum_{i=1}^{n}|b_i|\sum_{i=1}^{n}(|a_i|+|b_i|)^{p-1}$$

$$=\sum_{i=1}^{n}(|a_i|^p)^{1/p}\left(\left(\left(\sum_{i=1}^{n}|a_i|+|b_i|\right)\right)^{(p-1)}\right)^{1/q}$$

$$+\sum_{i=1}^{n}(|b_i|^{1/p}\left(\sum_{i=1}^{n}(|a_i|+|b_i|)^{p-1}\right)^{1/q}$$

$$=\sum_{i=1}^{n}\left((|a_i|+|b_i|)^{(p-1)q}\right)^{1/q}\left[\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n}|b_i|^p\right)^{1/p}\right]$$

we now that $1/p + 1/q = 1$

[by Holders theorem..

$$\frac{q+p}{pq} = 1$$

$$\Rightarrow q+p = pq$$

$$p = pq - q$$

$$p = q(p-1)$$

$$\frac{\left(\sum_{i=1}^{n} (|a_i| + |b_i|)\right)^p}{\left(\sum_{i=1}^{n} (|a_i| + |b_i|)^{(p-1)q}\right)^{1/q}} =$$

$$= \left(\sum_{j=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}$$

$$\Rightarrow \frac{\left(\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right)^q}{\left(\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right)^{1/q}}$$

$$= \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}$$

$$\left(\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right)^{1-1/q} = \left(\sum_{i=1}^{n} (|a_i|)^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}$$

$$\left(\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right)^{1/p} = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p} \rightarrow (2)$$

From (1) & (2)

we get.

Minkowski inequality:

$$\left(\sum_{i=1}^{n}(|ai|+|bi|)^{p}\right)^{1/p} \leq \left(\sum_{i=1}^{n}|ai|^{p}\right)^{1/p}+\left(\sum_{i=1}^{n}|bi|^{p}\right)^{1/p}$$

Hence the theorem.

Cauchy Schwartz inequality:-

If $p>1$ and q is such that $1/p + 1/q = 1$ then $\sum_{i=1}^{n} |ai|+|bi| \leq \left(\sum_{i=1}|ai|^{p}\right)^{1/p}$

$$\left(\sum_{i=1}^{n}|bi|^{q}\right)^{1/q}$$

If the condition $p-q=2$ sub in Holder's inequality. Then $\sum_{i=1}^{n} |ai|+|bi| \leq$

$$\left(\sum_{i=1}^{n}|ai|^{2}\right)^{1/2}, \left(\sum_{i=1}^{n}|bi|^{2}\right)^{1/2}$$

Hence the theorem.

Definition : Metric space:-

A Metric space is a non-empty. Let M together with a function $d: M \times M \rightarrow R$ statisfy the conditions

(i) If $d(x,y) \geq 0 \ \forall x,y \in M$

(ii) $d(x,y)=0$ iff $x=y \ \forall x,y \in M$

iii) $d(x,y)=d(y,x) \ \forall x,y \in M$

iv) $d(x,z) \leq d(x,y)+d(y,z) \ \forall x,y,z \in M$

The (iv) condition is a triangle inequality..

'd' is called a Metric (or) distance function. $d(x,y)$ is called.

Note:-

A Metric space with the Matric "d" is denoted by M.d.

Usual Metric:-

In R we define $d(x,y) = |x-y|$.

Then d is a Metric in R. It is called a usual Metric.

Ex!

1. In R we define $d(x,y) = |x-y|$. Then d is a Metric on R it is called a usual Metric.

Soln':

If R is defined $d(x,y) = |x-y|$

Clearly

$$d(x,y) = |x-y|$$

$$d(x,z) = |x-z|$$

(i) $d(x,y) \geq 0$

$$|x-y| \geq 0 \ \forall \ x,y \in M$$

(ii) $d(x,y) = 0$ Iff $x = -y$

$|x-y| = 0$

$x - y = 0$

$x = -y \ \forall x,y \in M$

iii) $d(x,y) = d(y,x) \ \forall x,y \in M$

$d(x,y) = |x-y|$

$= |y-x| \ \forall x,y \in M$

iv) $|x-z| \leq |x-y| + |y-z| \ \forall x,y,z \in M$

$d(x-y) \leq d(x,y) + d(y,z) \ \forall x,y,z \in M$

Hence d is Metric on R.

Note:-

If the complex number $z = x+iy$ is Idenity with an points x,y of the two dimension equivalent plane then the above distance formula taken the form

$$d(z,w) = \sqrt{(x-u)^2 + (y-v)^2}$$

Where $z = x+iy$ and $w = u+iv$

This is the usual distance between the points x,y and v,u to the plane

Discrete Metric Space:-

Any non-empty set M. We define d as follows. $d(x,y)$ then d is a Metric in M. It is called discrete M Metric space on M.

Ex:-

Any non-empty set M we define d as follows $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}$

Then

i) Clearly $d(x,y) \geq 0$ ($0 \leq 1$ a real number

ii) $d(x,y) = 0$ iff $x=y$ $d(x,y)=0$ if $x=y$

iii) $d(x,y) = d(y,x)$

$$d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}$$

$$d(y,x) = \begin{cases} 0 & \text{if } y=x \\ 1 & \text{if } y \neq x \end{cases}$$

$\therefore d(x,y) = d(y,x)$

iv) $d(x,z) \leq d(x,y) + d(y,z)$

$\therefore d(x,z) = 0$

case (i)

$x = z$

$d(x,z) =$

$d(x,y) + d(y,z) \geq d(x,z)$

case ii)

$d(x,z) = 1$

$x \neq z$

$d(x,y) + d(y,z) \geq d(x,z)$

$\therefore d(x,z) \geq d(y,z) \geq d(x,y)$

$d(x,z) \leq d(x,y) \leq d(y,z)$

Hence d is a Metric on M.

$\forall x,y,z \in M.$

Usual Metric R^n :-

In R^n we define $d(x,y) =$

$\left(\sum_{i=1}^{n} (xi - yi)\right)$ where $x = (xi, x_2 \ldots xn)$ and

$y = (y_1, y_2 \ldots yn)$. then d is a Metric on R^n

R is called the usual Metric on R^n

i) $d(x,y) \geq 0$

$$\left(\sum_{i=1}^{n} (xi - yi)^2\right)^{1/2} \geq 0$$

ii) $$\left(\sum_{i=1}^{n} (xp - yi)^2\right)^{1/2}$$

$\Leftrightarrow (xi - yi)^2$

$\Leftrightarrow (xi^2 - yi^2)$

$\Leftrightarrow (xi - yi)$

$\Leftrightarrow (x - y)$

iii) $$d(x,y) = \left(\sum_{i=1}^{n} (xi - yi)^2\right)^{1/2}$$

$$d(y,x) = \left(\sum_{i=1}^{n} (yi - xi)^2\right)^{1/2}$$

iv) $d(x,z)$

$$\left(\sum_{i=1}^{n} |ai + bi|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |ai|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |bi|^p\right)^{1/p}$$

$ai = xi - yi$; $bi = yi - zi$

$$\left(\sum_{i=1}^{n} |(xi - yi) + (yi - zi)|^2\right)^{1/2}$$

$$\left(\sum_{i=1}^{n} (xi - zi)^2\right)^{1/2} \quad \therefore (p \cdot q = 2)$$

Note:-

R^n with usual metric is called n dimensional Euclidean space.

Ex

Let $x, y \in R^2$. Then $x, x_1, x_2 \ldots x_n$ and $y, y_1, y_2 \ldots y_n$ where $x_1, x_2, y, y \in R$

we define $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

then d is a metric on R^2

proof:-

$$x, y \in R$$

$$x = x_1, x_2$$

$$y = y_1, y_2$$

we define $d(x,y) = |x_1, y_2| + |x_2 - y_2|$

(i) $|x_1 - y_1| + |x_2 - y_2| \geq 0$

(ii) $|x_1 - y_1| + |x_2 - y_2|$

$\Leftrightarrow x_1 - y_1 + x_2 - y_2 = 0$

$\Leftrightarrow x_1 = y_1$ and $x_2 = y_2$

$$x_1, x_2 = y_1, y_2$$

$$x = y$$

$$d(x,y) = d(y,x)$$

iii) $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

$$d(y,x) = |y_1 - x_1| + |y_2 - x_2| \; \forall x, y \in N$$

iv) $d(x,z) \leq d(x,y) + d(y,z)$

$d(x,z) = |x_1 - z_1| + |x_2 - z_2|$

$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

$d(y,z) = |y_1 - z_1| + |y_2 - z_2|$

$$d(x,y) + d(y,z) = |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2|$$

$= x_1 + x_2 - z_1 - z_2$

$= x_1 - z_1 + x_2 - z_2$

$= |x_1 - z_1| + |x_2 - z_2|$

$= d(x,z)$

$d(x,z) \leq d(x,y) + d(y,z) \; \forall \; x,y,z \in M.$

Hence proved.

When R^n we defined $d(x,y) = \text{Maximum}\{|x_i - y_i|\}$ $i = 1, 2, \dots n$ and $x = x_1, x_2 \dots x_n$ $y = y_1, y_2 \dots y_n$. Then d is a Metric on R^n

proof:-

(i) $d(x,y) \geq 0$

$\text{Maximum}\{|x_i - y_i|\} \geq 0$

ii) $d(x,y) = 0$ iff $x = 0$

$\text{Maximum}\{|x_i - y_i|\} = 0$

$\Leftrightarrow \text{Max}\{|x_i - y_i|\} = 0$

$\text{Max } x_i - y_i = 0$

$\text{Max } x_i = y_i$

$x = y.$

iii) $d(x,y) = d(y,x)$

$Max\{|x_i - y_i|\} = Max\{|y_i - x_i|\}$

Then $d(x,y) = d(y,x)$

iv) $d(x,z) \leq d(x,y) + d(y,z)$

$Max\{|x_i - z_i|\} \leq Max\{|x_i - y_i| + \cancel{Max}$
$Max\{|y_i - z_i|\}$

$Max\{|x_i - z_i|\} \leq Max\{|x_i - y_i + y_i - z_i|\}$

$Max\{|x_i - z_i|\} \leq Max\{|x_i - z_i|\}$

Hence d is a Metric on R^n

Ex:

Let M is ~~the~~ set of all bounded real numbers valued bounded define on a non - empty set defined

$d(f.g) = sup\{|f(x) - g(x)| / x \in E\}$

Then d is a Metric on M.

proof:-

we define $d(f,g) = sup\{|f(x) - g(x)|\}$

(i) $d(f,g) = sup\{|f(x) - g(x)| / x \in E\} \geq 0$

ii) $d(f,g) = 0 \Rightarrow sup\{|f(x) - g(x)|\} = 0$

$\Leftrightarrow |f(x) - g(x)| = 0 \ \forall \ x \in E$

$\Leftrightarrow f(x) - g(x) = 0$

$f(x) = g(x) = 0$

$\Leftrightarrow f(x) = g(x) \;\forall\; x \in E$

iii) $d(f, g) = \sup |f(x) - g(x)|$

$= \sup |g(x) - f(x)$

$= d(g \cdot f)$

iv) Let $f \cdot g \cdot h \in M$

we have $|f(x) - h(x)| = |f(x) - \cancel{g(x)} + \cancel{g(x)} - h(x)|$

$|f(x) - h(x)| \leq \sup |f(x) - g(x)| + \sup |g(x) - h(x)|$

$\sup |f(x) - h(x)| \leq \sup |f(x) - g(x)| +$

$\sup |g(x) - h(x)|$

$d(f, h) \leq d(f, g) + d(g, h)$

Hence d is a Metric on M.

11. Let l^α we note the set of all bounded sequence of real number $x = (x_n)$. Let $y = y_i \in l^\alpha$ defines the d on l^α $d(x, y) = l.u.b\, |x_n \cdot y_n|$. Then d is a Metric on l^α.

soln: We ~~now~~ denote defined $d(x, y) = l.u.b\, |x_n - y_n|$

(i) clearly $d(x, y) = l.u.b\, |x_n - y_n| \geq 0$,

(ii) $d(x, y) = 0 \Rightarrow l.u.b\, |x_n - y_n| = 0$

$\Leftrightarrow |x_n - y_n| = 0 \;\forall\; l \leq n < \alpha$

$\Leftrightarrow x_n - y_n = 0$

$\Leftrightarrow (x_n) = (y_n) \ \forall\ 1 \leq n \leq \infty$

$\Leftrightarrow \quad x = y$

iii) $d(x,y) = l.u.b\ |x_n - y_n|$

$= l.u.b\ |y_n - x_n|$

$= d(y,x)$

$d(x,y) = d(y,x)$

iv) Let $z = (z_n)$, $x = (x_n)$ $y = (y_n)$

$|x_n - z_n| = |x_n - y_n + y_n - z_n|$

$\leq |x_n - y_n| + |y_n - z_n|$

$l.u.b\ |x_n - z_n| \leq \{l.u.b\ |x_n - y_n|\} + \{l.u.b\ |y_n - z_n|\}$

$\therefore d(x,z) \leq d(x,y) + d(y,z)$

Hence d is a metric on l^∞.

12. Let M be the set of all sequence in R. Let $x, y \in M$ R. Let $x = (x_n)$; $y = (y_n)$ defined $d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n + y_n|)}$ then d is an metric on M.

proof: Let $x, y \in M$

To prove:

$d(x,y)$ is a real number ≥ 0

We have $\dfrac{|x_n - y_n|}{2^n(x+|x_n-y_n|)} \leq 1/2^n \ \forall n$

Also $\sum_{n=1}^{\infty} 1/2^n$ is a convergent series

[by comparision test)

$\therefore d(x,y)$ is a real number.

i) clearly $d(x,y) \geq 0$

ii) $d(x,y) = 0 \Rightarrow \sum_{n=1}^{\infty} \left\{ \dfrac{|x_n - y_n|}{2^n(1+|x_n-y_n|)} \right\} = 0$

$\Rightarrow |x_n - y_n| = 0 \ \forall n$

$\Rightarrow x_n - y_n = 0 \ \forall n$

$\Rightarrow x_n = y_n \ \forall n$

$\Rightarrow (x_n) = (y_n) = 0 \ \forall n$

$\Rightarrow x = y \ \forall n$

iii) $d(x,y) = \sum_{n=1}^{\infty} \dfrac{|x_n - y_n|}{2^n(1+|x_n-y_n|)}$

$= \sum_{n=1}^{\infty} \dfrac{|y_n - x_n|}{2^n(1+|y_n-x_n|)}$

$= d(y,x)$

$d(x,y) = d(y,x)$

iv) Let $x,y,z \in M$. Then.

$\dfrac{|x_n - z_n|}{1+|x_n-z_n|} = 1 - \dfrac{1}{1+|x_n-z_n|}$

$= 1 - \left\{ \dfrac{1}{1+|x_n - y_n + y_n - z_n|} \right.$

$$\leq 1 - \frac{1}{1+|x_n - y_n| + |y_n - z_n|}$$

$$\leq \frac{1 - |x_n - y_n| + |y_n - z_n| - 1}{1+|x_n - y_n| + |y_n - z_n|}$$

$$\leq \frac{|x_n - y_n|}{1+|x_n - y_n| + |y_n - z_n|} + \frac{|y_n - z_n|}{1+|x_n - y_n| + |y_n - z_n|}$$

$$\leq \frac{|x_n - y_n|}{1+|x_n - y_n|} + \frac{|y_n - z_n|}{1+|y_n - z_n|}$$

$$\frac{|x_n - z_n|}{1+|x_n - z_n|} \leq \frac{|x_n - y_n|}{1+|x_n - y_n|} + \frac{|y_n - z_n|}{1+|y_n - z_n|}$$

Both sides Multiplying inequality $1/2^n$ and talking the sum from $n=1$ to ∞.

$$\sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n|1+|x_n - z_n||} \leq \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n|1+|x_n - y_n||} + \sum_{n=1}^{\infty} \frac{|y_n - z_n|}{2^n|1+|x_n - y_n||}$$

$d(x,y) \leq d(x,y) + d(y,z)$

Hence d is a Metric on M.

problem: 1

Let d_1 and d_2 we two Matrices on M define $d(x,y) = d_1(x,y) + d_2(x,y)$

P.T d is a metric on M.

(i) $d(x,y) = d_1(x,y) + d_2(x,y) \geq 0$

(ii) $d(x,y) = 0 \Rightarrow d_1(x,y) + d_2(x,y) = 0$

$\Leftrightarrow d_1(x,y)$ and $d_2(x,y) = 0$

$\Leftrightarrow x = y$

iii) $d(x,y) = d_1(x,y) + d_2(x,y)$

$= d(y,x)$

iv) Let $x,y,z \in M$

$d_1(x,z) \leq d_1(x,y) + d_1(y,z) \rightarrow (1)$

$d_2(x,z) \leq d_2(x,y) + d_2(y,z) \rightarrow (2)$

$d_1(x,z) + d(x,z) \leq d_1(x,y) + d_2(y,z) +$
$d_2(x,y) + d_2(y,z)$

$d(x,z) \leq d(x,y) + d(y,z)$

Hence d is Metric on M.

2. Determine weather $d(x,y)$ define on R by $d(x,y) = (x-y)^2$ is a Metric (or) not.

Soln:-

Let $x,y \in R$

(i) $d(x,y) = (x-y)^2 \geq 0$ (ii) $d(x,y) = 0 \Leftrightarrow (x-y)^2 = 0$

$\Leftrightarrow x-y = 0$

$\Leftrightarrow x = y$

iii) $d(x,y) = (x+y)^2$

$= (y-x)^2$

$= d(y,x)$

iv) But the triangle inequality this not hold.

take $x=-5$ $y=-4$ $z=4$

$d(x,z) \leq d(x,y) + d(y,z)$

$d(x,z) = (-5-4)^2 = 81$

$d(x,y) = (-5+4) = 1$

$d(x,y) = (-4-4)^2 = 64$

$81 \geq 1+64$

$81 \geq 65.$

$\therefore$ d is not a Metric on R.

8. If d is a Metric on M. is d^2 not Metric on M.

soln:-

consider d(x,y) define on R by

$d(x,y) = |x-y|$

W.K.T

d is a Metric on M

$d^2(x,y) = |x-y|^2 = (x-y)^2$

But d^2 does not satisfying the condition triangle inequality.

Hence d^2 is not Metric.

If d is a Metric on M. PT $\sqrt{d}$ is Metric on M.

soln:- The given $d(x,y)$ is Metric on M.

(i) clearly $\sqrt{d(x,y)} \geq 0$

(ii) $\sqrt{d(x,y)} = 0 \Leftrightarrow |x-y|^{1/2} = 0$

$\Leftrightarrow x - y = 0$

$\Leftrightarrow x = y$

iii) $\sqrt{d(x,y)} = \sqrt{(x-y)}$

$= \sqrt{(y-x)}$

$= \sqrt{d(y,x)}$

iv) Let $x,y,z \in M$.

$d(x,z) = |x-z| = |x-y+y-z|$

$\sqrt{d(x,z)} = \sqrt{d(x,y) + d(y,z)}$

$\sqrt{d(x,z)} \leq \sqrt{d(x,y)} + d(y,z)$

Hence $\sqrt{d}$ is a Metric on M.

5. Let (M,d) Metric space $d(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$

P.T d is the Metric on M.

proof:- let (M,d) $d(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$

(i) $d(x,y) = \dfrac{d(x,y)}{1+|d(x,y)|} \geq 0$ $[\because d(x,y) \geq 0]$

ii) $d_1(x,y) = 0 \Rightarrow$

$\Leftrightarrow d(x,y) = 0$

$\Leftrightarrow x,y = 0$

$\Leftrightarrow x = y$

iii) $d_1(x,y) = \dfrac{d(x,y)}{1+d(x-y)}$

$= d_1(y,x)$

iv) Let $x,y,z \in M$

$$d_1(x,z) = \frac{d(x,z)}{1+d(x,z)}$$

$$= 1 - \frac{1}{1+d(x,z)}$$

$$= 1 - \frac{1}{1+d(x,y)+d(y,z)}$$

$$= \frac{1+d(x,y)+d(y,z)-1}{1+d(x,y)+d(y,z)}$$

$$= \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)}$$

$$\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$

$d_1(x,z) \leq d_1(x,y) + d_1(y,z)$

Hence d_1 is a metric on M.

Let (M, d) Metric space defined $d'(x,y) = \min\{1, d(x,y)\}$ P.T d' is a Metric on M.

proof:-

$$d'(x,y) = \min\{1, d(x,y)\}$$

(i) $d'(x,y) = \min\{1, d(x,y)\} \geq 0$

ii) $d'(x,y) = 0 \Rightarrow \min\{1, d(x,y)\} = 0$

$\Leftrightarrow d(x,y) = 0$

$\Leftrightarrow x = y$

iii) $d'(x,y) = \min\{1, d(x,y)\}$

$= \min\{1, d(y,x)\}$

$= d'(y,x)$

8. If $(M_1, d_1), (M_2, d_2) \ldots (M_n, d_n)$ are Metric spaces. Then $M_1 \times M_2 \times \ldots \times M_n$ is a Metric spaces with Metric d defined by $d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i)$, $x = (x_1, x_2 \ldots x_n)$ and $y = (y_1, y_2 \ldots y_n)$

proof:-

(i) $d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i) \geq 0$

(ii) $d(x,y) = 0 \Rightarrow \sum_{i=1}^{n} d_i(x_i, y_i) = 0$

$\Leftrightarrow x_i = y_i$

$\Leftrightarrow (x_1, x_2 \ldots x_n) = (y_1, y_2 \ldots y_n)$

$\Leftrightarrow x = y$

iii) $d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i)$

$= \sum_{i=1}^{n} d_i(y_i, x_i)$

$= d(y,x)$

$d(x,y) = d(y,x)$

iv) $d(x,z) = \sum_{i=1}^{n} (x_i, z_i)$

$= \sum_{i=1}^{n} \left[d_i(x_i, y_i) + d_i(y_i, z_i) \right]$

$\leq \sum_{i=1}^{n} d_i(x_i, y_i) + \sum_{i=1}^{n} d_i(y_i, z_i)$

$\leq d(x,y) + d(y,z)$

$d(x,z) \leq d(x,y) + d(y,z)$

Hence d is a Metric on M.

7. Let M be a non-empty set. Let $d: M \times M \to R$ be a function such that

(i) $d(x,y) = 0$ iff $x = y$

ii) $d(x,y) \leq d(x,z) + d(y,x)$ $\forall$ $x,y,z \in M$ prove that d is a Metric on M.

proof:-

put $y = x$ in (ii)

We have $d(x,x) \leq d(x,z) + d(y,x)$

$$0 \leq 2d(x,z)$$

$$d(x,z) \geq 0$$

To prove:-

$$d(x,y) = d(y,x)$$

put $z = x$ in (ii)

We get

$$d(x,y) \leq d(x,x) + d(y,x)$$

$$d(x,y) \leq 0 + d(y,x) \text{ [using (i)]}$$

Since this true $\forall\, x,y \in M$.

we have

$$\Rightarrow d(x,y) = d(y,x)$$

$$\Rightarrow d(y,x) \leq d(x,y)$$

Hence $d(x,y) = d(y,x)$

d is Metric on M.

9. In the Metric space (M,d) prove that $|d(x,x) - d(y,z)| \leq d(x,y)\ \forall\, x,y,z \in M$.

proof:-

Let $x,y,z \in M$

We have.

$$d(x,z) \leq d(x,y) + d(y,z) \rightarrow (1)$$

$$d(x,z) - d(y,z) \leq d(x,y) \rightarrow (2)$$

We Introducing x and y in (2)

$$d(y,z) - d(x,z) \leq d(x,y) \rightarrow (3)$$

From equ (2) and (3) we get

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

d is a Metric on M.

Bounded :-

Bounded set S in a Metric space :-

Let (M, d) be a Metric space

we say that a subset A of M is bounded.

If there exists a positive real number k such that

$$d(x, y) < k \; \forall \; x, y \in A$$

Diameter :-

Let (M, d) be a Metric space. Let $A \subseteq M$ then the diameter of A denoted by $d(A)$ is defined by

$$d(A) = \text{l.u.b} \{ d(x, y) / x, y \in A \}$$

Ex:

In R a diameter of any Interval equal to the length Interval

The diameter $[0, 1]$ is 1

Ex 3

In any Metric space $d(\phi) = -\infty$

Open ball (or) open sphere:

Let (M, d) be a Metric space let $a \in M$ and r be a positive real number. Then the open ball (or) open sphere is centre and radius r denoted by $B_d(a, r)$ is the subset of M given by $B_d(a, r) = \{ x \in M / d(a, x) < r \}$

When the Metric d under consideration is clear we write.

$B(a, r)$ instead of $B_d(a, r)$

Note: 1

$B(a, r)$ is a bounded set

Let $x, y \in B(a, r)$

They by defined of open ball

$$B(a, r) = \{x \in M / d(a, x) < r\}$$

$$d(a, x) < r \text{ and } d(a, y) \leq r$$

$$d(x, y) \leq d(x, a) + d(a, y)$$

$$< r + r$$

$$\leq 2r$$

$$d(x, y) < 2r$$

Ex 1

consider R with usual Metric. Let $a \in R$ prove that $B(a, r) = (a-r, a+r)$

proof:-

consider R with usual Metric.

Let $a \in R$

$$B(a, r) = \{x \in R / |a - x| < r\}$$

$$= \{x \in R / |\cancel{a} + x| < r\}$$

$$= \{x \in R \mid a - x < x < a + r\}$$

$$B(a, r) = (a - r, a + r)$$

Ex :2

consider C with usual Metric. let $A \in C$ then $B(a,r) = \{z \in C / |x - a| < r\}$

This is the interval of the circle write centre a an radius r.

Ex : 3

In R^2 with usual Metric $d(a,r)$ is the Interior of the circle with centre a and radius r.

Ex : 4

lett d be the discrete Metric on M

Then $d(a,r) = \begin{cases} M & \text{if } r > 1 \\ \{a\} & \text{if } r \leq 1 \end{cases}$

soln:-

The given condition d is Metric on M,

WKT,

$$d(a,x) = \begin{cases} 0 & \text{if } a = x \\ 1 & \text{if } a \neq x. \end{cases}$$

$$\therefore d(a,r) < M \rightarrow ①$$

$$B(a,r) = \{x \in M / d(x,y) < r\}$$

Case (i)

let $r < 1$ and $x \in M \rightarrow ②$

clearly every point $x \in M$

Such that $d(a,x) < r$.

$x = B(a,r) \rightarrow ③$

Form equ ① & ②

we get $M \subseteq B(a,r) \rightarrow ④$

eqn ① and ④

we get $M = B(a,r)$

If $r \le 1$ in this case for every point

$x \neq \infty$

clearly $d(a,x) < r \le 1$

$\Rightarrow d(a,x) < r \le 1$

$\Rightarrow d(a,x) < 1$

$\Rightarrow d(a,x) = 0$

$\Rightarrow d|a-x| = 0$

$\Rightarrow a = x$

$\therefore B(a,r) = \{a\}$

Hence $B(a,r) = \begin{cases} M & \text{if } r > 1 \\ \{a\} & \text{if } r \le 1 \end{cases}$

Open set :-

Let (M,d) be a Metric space. Let A be a subset of M. Then A is said to be open in A.

If for every $x \in A$. There exists the positive real number x such that $B(x,r) \le A$

Ex:1

In R is usual Metric $(0,1)$ is an open set.

proof :-

Let $x \in (0,1)$

Choose $r = \min\{x-0, 1-x\}$

$r = \min\{x, 1-x\}$

clearly $r > 0$.

$B(x, r) = (x-r, x+r) \subset A$

$\therefore B(x, r) \subset A$

Hence $(0,1)$ is an open set.

Ex: 2

In R is usual Metric $[0,1)$ is not open since no open ball with center with contains $[0,1)$

Ex: 3

Any open Interval (a,b) is an open in R with usual Metric.

proof:-

Let $x = (a,b)$

$r = \min\{x-a, b-x\}$

$B(x,r) = (x+r, x+r) \subseteq A.(a,b)$

Hence (a,b) is an open set.

Note:

Similarly we can prove that $(-\infty, a)$ and $(a, \infty]$ are open set.

Ex: 4.

In R with usual Metric of a set $\{0\}$ is not an open set.

proof:-

Since any open ball with center 0 with not contained in $\{0\}$.

Ex : 5

In R with usual Metric any Infinite non empty subset A(R) is not an opet set

proof :-

Any open ball in R is a bounded. open ball Interval which an Infinite subset of R. Hence It can be not contained. in the infinite subset A.

Hence A is not open in R.

Ex : 6.

Q is not open in R.

proof :-

Let $x \in Q$

Then for any $r > 0$ the interval $(x-r, x+r)$ contains both rational and irrational numbers.

$\therefore (x-r, x+r)$ is not a subset of a

Hence Q is not open in R.

Ex : 7.

Z is not open in R.

proof :-

Let $x \in Z$ then for any $r > 0$ the Interval $(x-r, x+r)$ is not a subset of Z.

Hence Z is not open in R.

Ex: 8.

The set of all irrational number.

proof:-

Let $x \in$ irrational numbers for any $r>0$ the interval $(x-r)(x+r)$ is not subset of irrational numbers.

Hence irrational number is not an open

Ex: 9

In a discrete Metric space M. every subset of A in open.

proof:-

To prove every subset A is open

case (i)

If $A = \phi$ trivialy A is open.

case ii)

If $A \neq \phi$. let $x \in A$ then $B(x_1 1/2) = \{x_i\} \in A$.

since in a discrete Metric.

$$B(a, r) = \begin{cases} M & \text{if } r > 1 \\ \{a\} & \text{if } r < 1 \end{cases}$$

Theorem: 2.1

In any Metric space M.

(i) ϕ is open

ii) M is open

proof:-

trivially empty set is open set.

ii) Let $x \in M$.

Clearly for ~~all~~ any $r>0$

$\therefore B(a,r) \subset M$.

Hence M is an open set.

Theorem : 2.2.

In any Metric space be (M,d) each open balls is a open set.

proof :-

Let $B(a,r)$ be a open ball in M.

Let $x \in B(a,r)$

Then $d(a,x) < r$.

$\therefore r - d(a,x) \geq 0$

Let $r_1 = r - d(a-x)$

To prove.

$B(a,r)$ is an open set

ie) To prove.

$B(x_1, r_1) = B(a,x)$

Let $y \in (x_1, r_1) \rightarrow ①$

$d(x,y) < r_1$

$< r - d(a,x)$

$d(x,y) + d(a,x) < r \rightarrow ①$

Now

$d(a,y) \leq d(a,x) + d(x,y)$

$\therefore d(a,y) < r$ [by (i)]

$y \in B(a,r)$

From eqn ① & 2.

$B(x_1, r_1) \subset B(a,r)$

Hence $B(a,r)$ is open set.

Thorem: 2.4

In any Metric space of interssection of finite number of open set is open.

proof:-

Let (M, d) be a Metric space

Let $A_1, A_2 \ldots A_n$ be a open set in M.

Let $A = A_1, A_2 \ldots \ldots A_n$

If $A = \phi$.

then A is open.

$A \neq \phi$

Let $x \in A$ then $x \in A_i$ for each $i = 1, 2, 3 \ldots$

Since each A_i is an open set.

There is a positive real number r such that.

$B(x, r_i) \subset A_i \rightarrow (1)$

Let $r = \min\{r_1, r_2, \ldots r_n\}$ obviously r is a positive real number.

an $B(x, r) \subseteq B(x, r_i) \ \forall \ i = 1, 2, 3 \ldots$

Hence $B(x, r) \subseteq A_i \ (i = 1, 2, 3 \ldots n)$

$\therefore B(x, r) \subseteq \bigcap_{i=1} A_i$

Hence A is open set.

For example.

consider R is usual Metric.

Let $A_n = (-1/n, 1/n)$ Then A_n is open in R $\forall n$.

But $\bigcap_{n=1}^{\infty} A_n = \{0\}$ which is open in R.

Equivalent Metric:-

Let d and ρ be the two Metric on M. Then the Metrices B and ρ are set to be Equivalent.

If open sets of (M, ρ) and open sets of (M, d)

6. Let (M, d) be a Metric space define $\rho(x,y) = 2d(x,y)$. then d and ρ are equivalent Metric.

soln:-

We know that ρ is the Metric on M.

We first p.T

$B_d(a, r) = B_\rho(a, 2r)$

Let $x \in B_d(a, r) \rightarrow$ (i)

$\Rightarrow d(a, x) < r$

$\Rightarrow 2d(a, x) < 2r$

$\Rightarrow \rho(a, x) < 2r$

$\Rightarrow x \in B_\rho(a, 2r) \rightarrow$ (ii)

ii) from eqn. (i) & (ii)

We get.

$B_d(a, r) \subseteq B_\rho(a, 2r) \rightarrow$ (iii)

Let $x \in B_\rho(a, 2r) \rightarrow$ (iv)

$\Rightarrow \rho(a, x) < 2r$

Multiplying on both Sides.

$\Rightarrow 2d(a, x) < 2r$.

$\Rightarrow d(a, x) < r$

$\Rightarrow x \in B_d(a, r) \rightarrow$ (v)

From (iv) & (v)

we get $B_P(a, 2r) \subseteq B_d(a, r) \rightarrow$ (vi)

From (iii) & (vi)

we get $B_d(a, r) = B_P(a, 2r) \rightarrow$ (vii)

Now let

G be any open ~~students~~ subsets (M, d)

Let ~~G~~ $a \in G$

Hence there exists $r > 0$ such that

$B_d(a, r) \subseteq G$ (by equ vii)

There G is open in (M, P)

Converse part:-

suppose G is open in (M, P)

Let $a \in G$.

Hence there exists r such that $B_d(a, r) \subseteq G$

$B_d(a, 1/2r) \subseteq G \rightarrow$ (by vii)

G is open in ~~[illegible]~~ (M, d)

and P are equivalent Matric on M.

Ex : 1

Let $M = R$ and $M_1 = [0,1]$

Soln:-

Let $A_1 = [0, 1/2]$

$A_1 = A \cap M_1$

Now, $A_1 = [0, 1/2] = (-1/2, 1/2) \cap (0,1]$

and

$(-1/2, 1/2)$ is open in R.

$[0, 1/2]$ is open in R $[0,1]$

Ex, : 2

$M = R$ and $M_1 = R$ and $M_1 = [1,2] \cup [3,4]$

Soln:-

Let $A_1 = [1,2]$

$A_1 = [1,2]$

$= \cap M.$

$\therefore [1,2]$ is open in M.

IIIly

3,4 is open in M_1.

Problem: 1

Let M_1 be a subspace by M. prove that every open set is open M. iff M_1 is itself is open in M.

Soln:-

Direct part!

Suppose every open set A_1 of M_1 is open.

Now, M_1 is open in M_1

Hence M_1 is open in M. $[M_1 \subset M]$

Convert part:-

Suppose M_1 is open M.

A_1 be a open set in M.

To prove,

A_1 is open set in M_1.

[by theorem 2.4]

Let A_1 be an open set in M iff M_1 is itself since $A_1 = A \cap M_1$ since A and M_1 are open in M.

A is open in M_1. $[M_1 \subseteq M_n]$

Interial of a set :-

Let (M, d) be a Metric space. Let $A \subset M$ Let $x \in A$. Then x is said to be as interior point of A. If there exists a positive real number such that

$B(x, r) \subseteq A$

The set of an interior points of A is called interior of A.

And it is denoted by Int A.

Note:-

Int $A \subseteq A$.

closed set :-

Let (M, d) be a Metric space Let $A \subset M$. Then A is said to be closed in M. If the complement A is open in M.

Ex: 1

~~Let~~ In R with usual Metric any closed Interval $[a, b]$ is closed set.

proof:-

To prove A is closed.

$$A^c = R - [a, b]$$

$$A^c = (-\infty, a) \cup (b, \infty)$$

Also $(-\infty, a)$ and (b, ∞) are open in R.

$\therefore$ A is closed.

A^c is open in R

Hence $A = [a, b]$ is closed in R.

Ex: 2.

with A is usual metric $[a, b]$ is neither closed nor open.

proof:-

$[a, b)$ is not open in R.

since A is not an interior point of $[a, b)$

Now $[a, b]^c = R - [a, b)$

$= (-\infty, a) \cup (b, \infty)$

and this set is not open.

since b is not an interior point

$\therefore [a, b)$ is not closed in R.

Hence $[a, b)$ is neither open nor closed.

Ex: 3

If R is usual Metric $(a,b]$ is neither closed nor open.

proof:-

$(a,b]$ is not open in R.

Since a is not ~~open~~ ~~in R.~~ an interior point of $(a,b]$

Now $(a,b]^c = R - (a,b]$

$= (-\infty, a) \cup (b, \infty)$

This set is not open.

a is not an interior point

$(a,b]$ is neither closed nor open.

Ex: 4.

x is closed.

proof:-

Let x^c is open

$$Z^c = \bigcup_{n=-\infty} (n, n+1)$$

Union open set is open

Z^c open

Z is closed.

Ex: 5

Q is not closed in R.

proof:-

Q^c = The set of all irrational number.

which is not open in R

Q is not closed in R.

Ex: 5

The set of all irrational number is not closed in R

proof:-

The set of all irrational number

$\{\text{irrational number}\}^c = Q$

Q is not open.

{irrational number} not closed R.

Ex: 6.

The set of all irrational number is not closed in R.

proof:-

Ex: 7

In R which usual Metric any $\{a\}$

proof:-

let $a \in R$

$\{a\}^c \neq \phi$

$\{a\}^c = (-\infty, a) \cup (\infty, b)$

also $(-\infty, a)$ and (a, ∞) are both open sets.

$(-\infty, a) \cup (a, \infty)$ is open.

$\{a\}^c = \phi$ is open in R.

$\therefore \{a\}$ is closed.

Ex : 8

Every subset of a discrete Metric is space is closed.

proof:-

Let (M,d) be a discrete Metric space

Let $A \subseteq M$.

since Every subset of a discrese Metric space is open.

$\therefore A^c$ is also a subset of discrete Metric space.

$\therefore A^c$ is open set.

A is closed.

Closed Ball or closed sphere:

Let (M,d) be a Metric space

Let $a \in M$

Let r be any positive real number

Then the closed ball (or) closed sphere with centre a and radius r and denoted by.

$B_d[a,r]$ defined By

$B_d[a,r] = \{x \in M / d(a,x) \leq r\}$ then the Metric space d under the considerations we write.

$[B_d]$ $B_d[a,r]$ initiated of $B_d[a,r]$

Ex : 1

In usual

In R is usual Metric $B[a,r]$ i.e)

$[a-r, a+r]$

Ex: 2

R^2 with usual Metric $a(a_1, a_2) \in R^2$

then $B[a,r] = \{x,y \in M / d(a_1,a_2)(x,y) \leq r^2$

$= \{x,y \in M / d(x-a_1)^2 (y-a_2)^2 \leq r\}$

Hence $B[a,r]$ is set of all points which (ie) within and on the airumstance of the cricle with center a and radius r.

Theorem:- 2.8.

In any Metric space every closed let ball is closed.

proof!-

Let (M,d) be a Metric space and.

Let $B[a,r]$ be a closed ball in M.

ie) To prove that.

$B[a,r]$ is closed ball

$\therefore B[a,r]^c$ is open.

case(i):

If $B[a,r]^c = \phi$

$\Rightarrow B[a,r]^c$ is open

$B[a,r]$ is closed.

case ii)

If $B[a,r]^c = \phi$

$x \in B[a,r]^c$

$x \notin B[a,r]^c$

$\Rightarrow d(a,x) \geq r$

$\Rightarrow d(a,x) - r > 0$

Let $r_1 = d(a,x) - r \rightarrow (1)$

To prove

$B[a,r] \subseteq B[a,r]^c$

$\Rightarrow y \in B[a,r] \rightarrow (i)$

$\Rightarrow d(x,y) < r_1 = d(a,x) - r$

$d(x,y) < (a,x) - r \rightarrow (2)$

ie)

$d(a,x) > d(x,y) + r$

now,

$d(x,z) \leq d(x,y) + d(y,z)$

$d(a,x) \leq d(a,y) + d(y,x)$

$d(a,x) - d(ay,x) \leq d(a,y)$

ie) $d(a,y) \geq d(a,x) - d(y-x)$

$d(a,y) \geq d(x,y) + r - d(y,-x)$

(From equ 2)

$d(a,y) > r$

$y \notin B[a,r]$

$y \in B[a,r]^c \Rightarrow$ ii)

From (i) and (ii)

we get.

$B[x_1, r_1] \subset B[a_1, r]^c$.

$\therefore B[a_1, r]^c$ is open in M.

Hence $B[a_1, r]$ is closed in M.

Hence the theorem.

Theorem: 2.9.

In any Metric space M.

(i) ϕ is closed.

(ii) M is closed.

proof:-

since $M^c = \phi$ is open.

M is closed.

I^{ly} $\phi^c = M$ is open

Hence ϕ is closed.

Note:

In any space (M, d) are both open set

Theorem: 2.10.

In any Metric space arbitrary Intersection of a closed set is closed

proof:-

Let M, d be a metric space and let $\phi\{A_i / i \in I\}$ be a collection of closed set.

To prove.

$\cap A_i$ is closed.

$(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c$ [by de-Morgon's Law]

A_i is closed

$(A_i)^c$ is open

Hence $\bigcup_{i \in I} A_i^c$ is open.

$(\bigcap_{i \in I} A_i)^c$ is open

$\bigcap_{i \in I} A_i$ is closed.

Theorem : 2.11.

Any Metric space the union of a finite number of closed set is closed.

proof:-

Let M_d B a Metric space

Let $A_1, A_2 \ldots A_n$ be a closed set M

To prove.

$A_1 \cup A_2 \cup \ldots \cup A_n$ is closed.

$(A_1 \cup A_2 \cup \ldots \cup A_n)^c = A_1^c \cap A_2^c \cap \ldots \cap A_n^c$

[by De-Morgon's Law]

since each A_i is closed.

A_i^c is open.

$A_1^c \cap A_2^c \cap \ldots \cap A_n^c$ is open

[by theorem 2.4]

$\therefore (A_1 \cup A_2 \cup \ldots \cup A_n)^c$ is open

$(A_1 \cup A_2 \cup \ldots \cup A_n$ is closed.

Note!-

The union of an infinite collection of closed set need not be closed.

Ex!-

consider R with the usual metric

Let $A_n = [1/n, 1]$ where $n = 1, 2, 3 \ldots \ldots$

$$\bigcup_{n=1}^{\infty} = \bigcup_{n=1}^{\infty} [1/n, 1]$$

$$= \{1\} \cup [1/2, 1] \cup [1/3, 1] \cup \ldots$$

$$= [0, 1]$$ which is not closed in R.

$\therefore \bigcup_{n=1}^{\infty} A_n$ is not closed.

Theorem : 2.12.

Let M be a metric space and M_1 be a subspace of M. Let $F_1 \subseteq M_1$. Then F_1 is closed in M_1. Iff there exists a set is which is closed in M. such that $F_1 = F \cap M$.

Proof!-

Let f_1 be closed in M_1

F_1^c is open in M.

$M_1 - F_1$ is open in M.

$\therefore M_1 - F_1 = A \cap M_1$

(where A is open in M.) [by theorem 2.6]

Direct part:-

$F_1 = F \cap M_1$ where F_1 is closed.

F_1 is closed in M_1

$M_1 - F_1$ is open in M_1

M_1/F_1 ~~is open/in~~ $M_1 - F_1 = A \cap M_1$

$$-F_1 = A \cap M_1 - M_1$$

$$F_1 = M_1 - A \cap M_1$$

$$F_1 = M_1 - A \quad (\because A \cap M_1 = A)$$

$$= A^c \cap M_1$$

A is open in M.

$\therefore A^c$ is closed in M.

$F_1 = F \cap M_1$ where $F = A^c$ is closed in M_1

proof of the converse is similar.

Closure :-

Let a be a subset of a Metric space (M, d). The closure of A denoted by $\bar{A}$ is defined to be the Intersection of all closed set which containing A.

Thus $\bar{A} = \cap \{B / B$ is closed in M & $A \subseteq B\}$

Theorem : 2.13

A is closed iff $A = \bar{A}$

proof:-

suppose $A = \bar{A}$

Direct part:-

since $\bar{A}$ is closed

A is closed.

Converse part:-

Suppose A is closed.

Then the smallest closed set containing A is A itself.

Note:-

In particular (i) $\phi = \bar{\phi}$

(ii) $M = \bar{M}$

Ex: 1

Consider R with usual metric

Proof:-

Let $A = [0,1]$

W.K.T

A is closed set.

$\therefore A = \bar{A}$ (ie) $[0,1]$

(ii) Let $A = (0,1)$

Then $[0,1]$ is a closed set containing $(0,1)$

obviously $[0,1]$ is the smallest closed set containing $[0,1]$

$\bar{A} = [0,1]$

Ex: 2.

In a discrete Metric space (M,d) any subset A of M is closed.

Proof:-

Let M, d be Metric space

Let $A \subseteq M$

Every subset of a discrete Metric space is open - n.

A^c is also a subset of a discrete Metric space

A^c is open

Hence A is closed

Hence $A = \bar{A}$

Theorem: 2.14.

Let (M, d) be a Metric space. Let $A, B \subseteq M$. Then.

(i) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

ii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

iii) $\overline{A \cap B} = \bar{A} \cap \bar{B}$

proof:-

(i) Let $A \subseteq B$

Now $\bar{B} \supseteq B \supseteq A$

$\bar{B}$ is the closed set containing in A

But $\bar{A}$ is the smallest closed set containing in A.

$\bar{A} \subseteq \bar{B}$

(ii) we have $A \subseteq A \cup B$

$\bar{A} \subseteq \overline{A \cup B}$ (by (i)) $\rightarrow$ (1)

$||^{ly}$ $B \subseteq A \cup B$

$\bar{B} \subseteq \overline{A \cup B}$ (by i) $\rightarrow$ (2)

Equ (1) & (2)

$\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$ $\rightarrow$ (3)

$\bar{A}$ is closed set containing A.

$\bar{B}$ is closed set containing B.

$\bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$

But $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$

ie) $A \cup B \subseteq \overline{A \cup B}$

$$\overline{A \cup B} \subseteq \bar{A} \cup \bar{B} \rightarrow (4)$$

From (3) & (4) we get.

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

iii) We have

$$A \cap B \subseteq A$$

$$\overline{A \cap B} \subseteq \bar{A} \quad (\text{by (i)})$$

lly

$$A \cap B \subseteq A$$

$$\overline{A \cap B} \subseteq \bar{B}$$

$$\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$$

Dense (or) Every where dense:-

A subset A of a Metric space M. is said to be dense in M. (or) every where dence If $\bar{A} = M$.

Separable:-

A Metric space M is said to be separable if there exists a countable dense subsen in M.

Interior of a set :-

Theorem : 2.17

Let (M, d) be a Metric space set $A, B \subset M$.

(i) A is open iff $A = \text{Int } A$.

In particular $\text{Int } \phi = \phi$ and $\text{Int } M = M$.

ii) $\text{Int } A$ = union of all open set contained in A.

iii) $\text{Int } A$ is an open subset of A and If B is anyother open set containing A. Then $B \subseteq \text{Int } A$.

iv) $A \subseteq B \Rightarrow \text{Int } A \subseteq \text{Int } B$

v) $\text{Int } (A \cap B) = \text{Int } A \cap \text{Int } B$

(vi) $\text{Int } (A \cup B) = \text{Int } A \cup \text{Int } B$.

proof:-

(i) It follows from defn of open set.

(ii) Let $G = \cup \{B / B$ is an open subset of $A\}$

To prove.

$\text{Int } A = G$

Let $x \in \text{Int } A$.

$\therefore$ there exist a positive real number r.

such that $B(x, r) \subseteq A$.

Thus $B(x, r)$ is open set contained in A.

$B(x, r) \subseteq G$.

$x \in G$.

interior of $A \subseteq B$ $\to$ ①

Let $x \in G$.

Then there exists an open set B

Such that $x \in B$ and $B \subseteq A$.

since B is an open set and $x \in B$ there

exists the positive real number r.

Such that $B(x, r) \subseteq B \subseteq A$.

x is an interior points of M.

Hence $G \subseteq$ Int A $\to$ ②

From ① & ② we get.

$G =$ Int A.

iii) since union of any collection of open sets in open. (by (ii))

$\Rightarrow$ Int A is an open set

Trivially. Int $A \subseteq A$

Let B be any open set contains in A.

Then $B \subseteq G =$ Int A (by (ii))

Int A is the Largest open set contained in B

$B \subseteq$ Int A.

iv) Let $x \in$ Int A.

$\therefore$ there exists the real number $r > 0$ such that.

$B \cup A \subseteq A$.

Since $A \subseteq B$

Hence $B(x, r) \subseteq B$

$x \in$ Int B.

Hence Int $A \subseteq$ Int B.

(V) $A \cap B \subseteq A$

$\text{Int } A \cap B \subseteq \text{Int } A$

$||^{ly}$

$A \cap B \subseteq B$

$\text{Int } A \cap B \subseteq \text{Int } B$

$\text{Int } A \cap B \subseteq \text{Int } A \cap \text{Int } B.$

now.

$\text{Int } A \subseteq A$

$\text{Int } B \subseteq B$

Hence $\text{Int } A \cap \text{Int } B \subseteq A \cap B \rightarrow (1)$

Thus $\text{Int } A \cap \text{Int } B$ is an open set contained $A \cap B$.

But $\text{Int } (A \cap B)$ is the largest open set contained $A \cap B$.

$\text{Int } A \cap \text{Int } B \subseteq \text{Int } A \cap B \rightarrow (2)$

From equ (1) & (2) we get.

$\text{Int } (A \cap B) = \text{Int } A \cap \text{Int } B$

$\text{Int } (A \cup B) = \text{Int } A \cup \text{Int } B$

$A \subseteq A \cup B$

~~Int (AUB)~~

$\text{Int } A \subseteq \text{Int } A \cup B$

$\text{Int } B \subseteq \text{Int } A \cup B$

$\text{Int } A \cup \text{Int } B \subseteq \text{Int } (A \cup B)$

$\text{Int } (A \cup B) \geq \text{Int } A \cup \text{Int } B.$

Limit point:-

Let (M, d) be a Metric space let $A \subseteq M$. Let $x \in M$ then x is called a limit point (or) cluster point accumulation point of A.

If every open ball with center x contains atleast one point of A difference from x let us.

$$B(x, r) \cap - \{x\} \neq \phi \quad \forall r > 0$$

Derived set:-

The set of all limite point of A. is called the derived set of A and it's denoted by $D(A)$

<u>Note</u> :(i)

(i) x is not a limit point of A Iff there exist an open ball $B(x, r)$ such that $B(x, r) \cap A - \{x\} = \phi$

(ii) Let (M, d) be a Metric space Let $A \subseteq B$ then x is a limit point of A iff each open ball with under x contains an infinite number of point of A.

iii) Any infinite ~~numbers of~~ subset of Metric space has no limit.

iv) Let M be a Metric space and $A \subset M$. Then $\lambda = A \cup D(A)$.

v) $x \in \bar{A}$ ~~closed~~ iff $B(x, r) \cap A \neq \phi \ \forall r > 0$

vi) A is closed iff A cantains all it is points.

UNIT - II

complete Metric space:-

Let (M, d) be a Metric space. Let $(x_n) = x_1, x_2 \ldots x_n$ be a sequence of points sin M. Let $x \in M$ we say that (x_n) convergence to x if given $\varepsilon > 0$ there exist the integer n_0 such that $d(x_n, x) < \varepsilon \ \forall \ n \geq n_0$ also x is called a limit. of (x_n). If (x_n) converges to x.

we write $\lim_{n \to \infty} (x_n) = x$ or $x_n \to x$.

cauchy sequence:-

Let (M, d) be a Metric space let (x_n) be a sequence of points in M (x_n) is said to be cauchy sequence in M. If given $\varepsilon > 0$ there exists a positive integer n_0 such $d(x_n, x_m) < \varepsilon \ \forall \ m, n \geq n_0$ every cauchy sequence is convergent.

complete:-

A Metric space M is said to be complete if every cauchy sequence in M converges to a point in M.

Theorem : 3.1

For a convergent sequence (x_n) the Limit is unique.

proof :-

Suppose $(x_n) \to x$ and $(x_n) \to y$

To prove

$$x = y$$

Let $\varepsilon > 0$ be given then there exists positive Integer's n_1, n_2 such that

$$d(x_n, x) < \varepsilon/2 \quad \forall \; n \geq n$$

$$d(x_m, y) < \varepsilon/2 \quad \forall \; n \geq n_2$$

Let m be the positive integer such that.

$$m \geq n_1, n_2$$

then
$$d(x, y) \leq d(x, x_m) + d(x_m, y)$$

$$d(x, y) < \varepsilon/2 + \varepsilon/2$$

$$< \cancel{2}\,\varepsilon/\cancel{2}$$

$$< \varepsilon$$

$$d(x, y) < \varepsilon$$

$$d(x, y) = 0$$

$$x - y = 0$$

$$x = y$$

Hence the limit is unique.

Theorem: 3.3.

Let (M, d) be a Metric space any convergence sequence in M is a cauchys sequence.

proof:-

Let (x_n) be a convergence sequence in M.

converging to $x \in M$.

Let $\epsilon > 0$ be given there exists positive Integer n_0 such that $d(x_n, x) < \epsilon/2 \ \forall n \geq n_0$

Now $d(x_m, x_n) \leq d(x_m, x) + d(x, x_n)$

$< \epsilon/2 + \epsilon/2 \ \forall n, m \geq n_0$

$< 2\epsilon/2$

$< \epsilon \ \forall n, m \geq n_0$

x_n is cauchy's sequence.

Note:-

The converse of above theorem is not true.

Ex:-

consider a metric space $(0, 1]$ with usual metric.

$(1/n)$ is a cauchy sequence in $(0, 1]$

but this sequence does not convergent to any point $(0, 1]$

Theorem: 3.2.

Let M be metric space and $A \subseteq M$.

Then,

(i) $x \in \bar{A}$ iff there exists a sequence (x_n) in A such that $(x_n) \to x$

(ii) x is a limit point of A iff there exists (x_n) of distinct points in A. such that $(x_n) \to x$.

proof:-

Direct part:-

let $x \in \bar{A}$

Then $x \in A \cup D(A)$ $(\because \bar{A} = A \cup D(A))$

$\therefore x \in A$ or $x \in D(A)$

case(i)

If $x \in A$ that the constant sequence $x_1, x_2 \ldots \ldots$ is a sequence in a converging in x

case (ii)

$x \in D(A)$

then x limit point of A.

The open ball $B(x, 1/n)$ contains Infinite number points of A.

We choose $x_n \in B[x, 1/n] \cap A$.

Such that.

$x_n \neq x_1, x_2 \ldots x_{n-1}$ for each n.

$\therefore (x_n)$ is a sequence of distinct.

$d(x_n, x) < 1/n \ \forall n$

$\lim_{n\to\infty} d(x_n, x) = 0$

$\therefore (x_n) \to x$

<u>Converse part</u>:-

Suppose there exists the sequence such that $(x_n) \to x$

To prove:

$x \in \bar{A}$

then for any $r>0$ there exists positive integer n_0 such that

$d(x_n, x) < r \ \forall n \geq n_0)$

$(\therefore x_n \in B(x, r)$

$x_n \in B(x, r) \cap A.$

$B(x, r) \cap A \neq \phi$

$x \in \bar{A}$

(ii) (i) <u>Direct part</u>:-

Suppose x is a limit point of A

$x \in D(A)$

$x \in A \cup D(A)$

$x \in \bar{A}$

$(x_n) \to x$ (by i)

(ii) <u>Converse part</u>:-

Suppose (x_n) is a sequence distinct points of A.

Then $B(x, r) \cap A$ is infinite.

$\therefore x \in D(A)$

Hence x is limit point of A.

Theorem : 2.17 (1st Unit continuous)

Let M be a Metric space and $A \subseteq M$

(i) then the following are equivalent

ii) The only ~~open~~ closed set which contains A to M.

iii) The only open set disjoints from A is empty.

iv) A Intersect every non-empty open set.

v) A intersect every open ball.

proof :-

(i) $\Rightarrow$ (ii)

Suppose A is dense in M.

Then $\bar{A} = \phi$

Now, let $F \subseteq M$ any closed set containing A

we have

Hence $M \subseteq F$

$M = F$

Then only closed set which contain is M.

(ii) $\Rightarrow$ (iii)

Suppose (iii) is not true

Then there exists a non-empty open set B such that $B \cap A = \phi$

$\bar{B}$ is a closed set and $\bar{B} \supseteq A$.

which is contradiction to (ii)

Hence (ii) $\Rightarrow$ (iii) obviously.

(iii) $\Rightarrow$ (iv)

(iv) $\Rightarrow$ (v)

Since every open ball is an open set.

(v) $\Rightarrow$ (i)

Let $x \in M$.

suppose every open ball $B(x,r) \cap A$

then by corollary (2) of theorem 2.16.

$x \in \bar{A}$

$M \subseteq \bar{A}$

By trivially $\bar{A} \subseteq M$.

$\bar{A} = M$

A is dense in M.

Theorem: 2.15

Let (M, d) be a metric space

Let $A \subseteq M$. Then x is a limit point of A iff each open ball with centre x contains an infinite no of points of A.

proof:-

Let x be a limit point of A.

suppose an open ball $B(x,r)$ $\angle$ only A finite no of.

Let $B(x,r) \cap (A - \{x\}) = \{x_1, x_2 \dots x_n\}$

Let $r_1 = \min\{d(x, x_i) / i = 1, 2, 3 \dots n\}$

since $x \neq x_i$ $d(x, x_i) > 0 \ \forall i = 1, 2, \dots n$

and hence $r_1 > 0$

Also $B(x, r_1) \cap (A - \{x\}) = \phi$.

x is a not Limit point of A.

Which is contradiction.

Every open ball with centre x.

Theorem: 2.16.

Let M be a Metric space and $A \subseteq M$. Then $\bar{A} = A \cup D(A)$

proof:-

$x \in A \cup D(A)$

We shall prove that.

$x \in \bar{A}$

suppose $x \notin \bar{A}$

$x \in M - \bar{A}$

and hence $\bar{A}$ is closed

$M - \bar{A}$ is open

There exists a open ball

$B(x, r)$ containing $M - \bar{A}$

$B(x, r) \cap \bar{A} = \phi$

$B(x, r) \cap A =$ [Since $A = \bar{A}$

$x \notin A \cup D(A)$

which is contradiction

$x \in \bar{A}$

$A \cup D(A) \subseteq \bar{A} \rightarrow (1)$

Now Let

$x \in \bar{A}$

To prove

$x \in A \cup D(A)$ if $x \in A$.

clearly $x \in A \cup D(A)$

suppose $x \notin D(A)$

Then there exists an open ball $B(x, r)$ such that.

$$B(x, r) \cap A = \phi$$

$$B(x, r)^c \supseteq A$$

And $B(x, r)^c$ is closed

But $\bar{A}$ is the smallest closed set containing A

$$\bar{A} \subset B(x, r)^c$$

But $x \in \bar{A}$ and $x \in B(x, r)^c$ which is a condradiction

Hence $x \in D(A)$

$$x \in A \cup D(A)$$

$$\bar{A} \subseteq A \cup D(A) \rightarrow (2)$$

From (1) & (2) we get.

$$\bar{A} = A \cup D(A)$$

Example for derived set:-

1. consider R with usual Metric

a) $A = [0, 1)$

proof:-

$$\bar{A} = A \cup D(A)$$

$$= [0, 1) \cup [0, 1]$$

$$= [0, 1]$$

b), Let $A = \{1, 1/2, \dots 1/n \dots\} \cup \{0\}$

$$\bar{A} = A \cup D(A)$$

Let $A = \{1, 1/2, \dots 1/n \dots\} \cup \{0\}$.

c) $\bar{Z} = Z \cup D(x)$

$= Z \cup \phi$

$= Z$

Z is closed.

D) $\bar{Q} = Q \cup D(A)$

$= Q \cup R$

$= R$

R is closed.

Q. To prove that $R \times R$ with usual metric.

$\overline{Q \times Q} = (Q \times Q) \cup D \cup (Q \times Q)$

$= (Q \times Q) \cup (R \times R)$

$= R \times R$

$= Q \times Q$ is not closed.

problem:

P.T for any subset A of a Metric space $d(A) = d(\bar{A})$ where $d(A)$ is the diameter of A.

soln:-

We have

A containing $\bar{A}$, $A \subset \bar{A}$

$d(A) = d(\bar{A}) \rightarrow ①$

Now, let $\varepsilon > 0$ be given

We claim that

$d(\bar{A}) \leq d(A) + \varepsilon$

let $x, y \in \bar{A}$

$B(x, 1/2 \varepsilon) \cap A \neq \phi$

and $B(y, 1/2 \varepsilon) \cap B \neq \phi$.

Let $x_1 \in B(x, 1/2\,\varepsilon) \cap A$.

and $x_2 \notin B(y, 1/2\,\varepsilon) \cap A$.

$d(x, x_1) < 1/2\,\varepsilon$ and

$d(y, x_2) < 1/2\,\varepsilon \rightarrow$ (2)

Also

$x_1 \in A$, and $x_2 \in A$

$\Rightarrow d(x_1, x_2) \leq d(A) \rightarrow$ (3)

Now $d(x, y) \leq d(x, x_1) + d(x, x_2) + d(x_2, y)$

$< 1/2\,\varepsilon + d(A) + 1/2\,\varepsilon$

[By eqn (2) & (3)]

$= d(A) + \varepsilon$

Thus $d(x, y) < d(A) + \varepsilon$

l.u.b $\{d(x, y) / x, y \in \bar{A}\} \leq d(A) + \varepsilon$

$d(\bar{A}) + d(A) + \varepsilon$

Now since ε is arbitrary

we have

$d(\bar{A}) \leq d(A) \rightarrow$ (4)

by equ (1) & (4) we get.

$d(A) = d(\bar{A})$

problem:-

A closed set E such that both E and E are dense in R.

Soln:-

Let $E = Q$.

Since any open ball $B(x, r) = (x-r)(x+r)$

contains both irrational.

$\mathbb{Q}$ and $\mathbb{Q}^c$

Hence $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$.

IInd Unit:-

Ex:-

$\mathbb{R}$ with usual Metric with $\mathbb{R}^n$

Ex:

Let $\mathbb{C}$ with usual Metric is complete.

Proof:-

(z_n) be a cauchy sequence in $\mathbb{C}$

Let $z = x + iy$

where $x_n, y_n \in \mathbb{R}$.

To prove.

(x_n) and ~~(yn)~~ (y_n) be a cauchy sequence

in $\mathbb{R}$.

Let $\varepsilon > 0$ be given

(z_n) is cauchy's sequence.

There exists a $^{+ve}$ Integer n_0

such that.

$d(z_m, z_n) < \varepsilon$

ie) $|z_n - z_m| < \varepsilon \ \forall\, n, m \geq n_0$

Now,

$|x_n - x_m| \leq |z_n - z_m|$

and

$|y_n - y_m| \leq |z_n - z_m|$

Hence $|x_n - x_m| < \varepsilon$ and

$|y_n - y_m| < \varepsilon \ \forall \ n, m \geq n_0$

$\therefore (x_n)$ & (y_n) are cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is complete.

There exists $x, y \in \mathbb{R}$ such that

$x_n \to x$ and $y_n \to y$.

Let $z = x + iy$

To prove $z_n \to z$

Now $|z_n - z| = |x_n + iy_n - x + iy|$

$= |x_n - x + i(y_n - y)|$

$= x_n - x + 1$

$= c.$

$\varepsilon > 0$ be given

Since $(x_n) \to x$: $y(n) \to y$.

There exists a positive integer n_1, n_2

such that $|x_n - x| < \varepsilon/2 \ \forall \ n \geq n_1$

$|y_n - y| < \varepsilon/2 \ \forall \ n \geq n_2$

Let $n_3 = \max\{n_1, n_2\}$ from the

equ we get

$|z_n - z| < \varepsilon/2 + \varepsilon/2$

$< \varepsilon \ \forall \ n \geq n_3$

$\therefore (z_n) \to z$

Hence C is a complete.

Ex:

In any discrete Metric space is complete

proof:-

Let (M, d) be a discrete Metric space (x_n) be cauchy sequence in M.

Then there exists a $+^{ve}$ Integer n_0 such that

$$d(x_n, x_m) < \varepsilon \ \forall\ n, m \geq n_0$$

Since d is a discrete Metric distance that been any x points either zor (or) 1.

$$d(x_m, x_n) = 0 \ \forall\ m, n \geq n_0$$

$$x_n = x_{n_0} = x \ \forall\ n \geq n_0$$

$$d(x_n, x) < \varepsilon$$

$$d(x_n, x) = 0 \ \forall\ n \geq n_0$$

$$(x_n) \to x$$

Hence m is complete.

Ex: 4.

R^n with usual metric is complete.

proof:-

Let (x_p) be a cauchy sequence in R^n

Let $(x_p) = (x_{p_1}, x_{p_2} \cdots x_{p_n})$

$\varepsilon > 0$ be given.

Then there exists a $+^{ve}$ Integer n_0 such that.

$$d(x_p - x_q) < \varepsilon \ \forall\ p, q \geq n_0$$

$$\left[\sum_{k=1}^{n}(x_{pk}-y_{qk})^2\right]^{1/2} < \varepsilon \ \forall\ p,q \geq n_0$$

squaring on both sides.

$$\left[\sum_{k=1}^{n}(x_{pk}-y_{qk})^2\right] < \varepsilon^2 \ \forall\ p,q \geq n_0$$

for each $k = 1, 2, 3 \ldots n$

we now,

$$|x_{pk} - y_{qk}| < \varepsilon \ \forall\ p,q \geq n_0$$

$\therefore$ x_{pk} is a cauchy sequence in R.

for each $k = 1, 2, \ldots n$

Since R is complete.

There exists.

Let $y_1, y_2 \ldots y_n$

To prove $(x_p) \to y$

since $(x_{pk}) \to y_k$ there exists the Integer M_k such that $|x_{pk} - y_k| < \varepsilon/\sqrt{n}$

$\forall\ p \geq M_k$

$$|x_{pk} - y_k| < \varepsilon/\sqrt{n}$$

Let n_0 max $\{m_1, m_2 \ldots m_n\}$

then $p(x_p, y) = \left[\sum_{k=1}^{n} |x_{pk} - y_k|^2\right]^{1/2}$

$< n^{1/2}$

$$\left[\sum |\varepsilon/\sqrt{2})^2\right]^{1/2} \ \forall\ p \geq M_0$$

$(x_p \to y.$

Hence R^n is complete.

Ex: 5

l_2 is complete.

proof:-

let (x_p) be a cauchy sequence in l_2.

Let $(x_p) = (x_{p_1}, x_{p_2} \dots x_{p_n})$

let $x > 0$ be given there exists +ve Integer n_0 such that.

$$d(x_p\ y_q) < \varepsilon \ \forall\ p, q \geq n_0$$

(ie) $\sum_{n=1}^{n} |x_{pn} - y_{qn}| < \varepsilon$

squaring on both sides.

$$\sum_{n=1}^{\infty} (x_{pn} - y_{qn})^2 < \varepsilon^2 \ \forall\ p, q \geq n_0 \rightarrow (1)$$

For each $n = 1, 2 \dots$

we now

$|x_{pn} - y_{qn}| < \varepsilon \ \forall\ p, q \geq n_0$

$\therefore (x_{pn}$ is a cauchy sequence in $R)$

Since R is complete.

There exists $y_n \in R$ such that

$$(x_{pn}) \rightarrow y_n \rightarrow (2)$$

Let $y = y_1, y_2 \dots y_n$

To prove $y = l_2$ and $(x_p) \rightarrow y$

for fixed +ve Integer M.

we have.

$$\sum_{n=1}^{M} |x_{pn} - x_{qn}|^2 < \varepsilon^2 \ \forall\ p, q \geq n_0$$

[using equ (1)]

Fixing q and along $p \to \infty$ in this finite sum we get.

$$\sum_{n=1}^{M} (y_n - x_{qn})^2 < \varepsilon^2 \ \forall\ q \geq n_0$$

Since this is true for every +ve Integer n.

$$\sum_{n=1}^{\infty} (y_n - x_{qn})^2 < \varepsilon^2 \ \forall\ q \geq n_0 \rightarrow (3)$$

$$= \left(\sum_{n=1}^{\infty} |y_n - x_{qn} + x_{qn}|^2\right)^{1/2}$$

$$= \left(\sum_{n=1}^{\infty} |y_n - x_{qn}|^2\right)^{1/2} + \left(\sum_{n=1}^{\infty} |x_{qn}|^2\right)^{1/2}$$

$$< \varepsilon + \left(\sum_{n=1}^{\infty} |x_{qn}|^2\right)^{1/2} \text{ [by eqn (3)]}$$

for $q \geq n_0$

Since $x_q \in l_2$

we have

$$\left(\sum_{n=1}^{\infty} |x_{qn}|^2\right)^{1/2} \text{ is convergent.}$$

$\therefore$ $y \in l_2$ also equ 2 gives.

$d(y, x_q) < \varepsilon \ \forall\ q > n_0$

$|||^{ly}$ $d(y, x_p) < \varepsilon \ \forall\ p \geq n_0$

$\therefore$ $(x_p) \to y$

Hence l_2 is complete.

Theorem: 3.4.

A subset of a complete Matrices Spaces M is complete iff A is closed.

proof:-

let $A \subseteq M$.

Direct part:-

Suppose M is complete.

To prove that A is closed.

It is enough to prove that contains all its Limit point.

Let x be a limite point of A.

[by theorem: 3.2 (ii) statement]

x is a limit point of $A \Leftrightarrow (x_n) \to x$

There exists a sequence of $(x_n) \to x$

Since A is complete

$\therefore x \in A$.

A contains all its Limit point

Hence A is closed.

part II

converse part:-

A is closed

To prove that

M is complete.

Let A be a closed subset of $M A (\subseteq M)$

Let (x_n) be a cauchy sequence in M.

Since M is complete's

There exists $x \in M$ such that.

$(x_n) \to x$ does (x_n) is a sequence A converging to x.

$\therefore x \in \bar{A}$ [by theorem 3.2(i) write the page]

since A is closed.

WKT

A is closed $\Leftrightarrow A = \bar{A}$

$x \in A$

Thus every cauchy sequence of (x_n) in A converges to a point $x \in A$.

A is complete.

since $A \subseteq M$.

M is complete.

problem: 1

Let A, B be a subset of R. Prove that $\overline{A \times B} = \bar{A} \times \bar{B}$

proof:-

ie) To prove that.

i) $\overline{A \times B} \subseteq \bar{A} \times \bar{B}$

ii) $\bar{A} \times \bar{B} \subseteq \overline{A \times B}$

Let $(x, y) \in \overline{A \times B} \rightarrow ①$

There exists the sequence $(x, y) \in \bar{A} \times \bar{B}$ such that $(x_n, y_n) \rightarrow (x, y)$ [By theorem 3.2(i)]

$(x_n) \rightarrow x$ and $(y_n) \rightarrow y$.

Also (x_n) is a sequence in A and (y_n) is a sequence in B

$x \in \bar{A}$ and $y \in \bar{B}$ [By theorem 3.2]

$(x, y) \in \bar{A} \times \bar{B} \rightarrow ②$

From ① & ② we get.

$\overline{A \times B} \subseteq \overline{A} \times \overline{B} \rightarrow$ ③

(ii) Let $(x, y) \in \overline{A} \times \overline{B} \rightarrow$ ④

$x \in \overline{A}, y \in \overline{B}$

There exist the sequence (x_n) in and sequence of (y_n) in B such that

$(x_n) \rightarrow x$ and $(y_n) \rightarrow y$

(x_n, y_n) is a sequence in $A \times B$

$\overline{A} \times \overline{B}$ which convergents to (x, y)

$\therefore (x, y) \in \overline{A \times B} \rightarrow$ ⑤ [By theorem 3.8(i)]

From ④ & ⑤ we get.

$\overline{A} \times \overline{B} \subseteq \overline{A \times B} \rightarrow$ ⑥

From.

$\overline{A} \times \overline{B} \subseteq \overline{A \times B}$

2. If A and B are closed subset of R_1. prove that $A \times B$ is a closed subset in $R \times R$.

proof :-

Let A and B are closed set.

we have $A = \overline{A}$ and $B = \overline{B}$

By using previous theorem $\overline{A \times B} = \overline{A} \times \overline{B}$

$= A \times B$

$A \times B$ is a closed set

No where dense :-

A subset of a metric space M. is said to be no where dence in M.

In interior of $\overline{A}$ = ϕ

First category:-

A subset n of a Metric space M is said to be first category in A.

If A can be expressed as a countable union of no where dense set.

Second category:-

A set which is not of first category it is called a second category.

Theorem: 3.5

Candor's Intersection of theorem:-

Statement:-

let M be a Metric space and M is complot [Every cauchy sequence is convergence] iff every sequence F_n of a non-empty closed sub set of M such that $F_1 \geq F_2 \geq F_3 \geq \cdots \geq F_n \geq$ and $d(F_n) \to 0$ Then $\bigcap_{n=1}^{\infty} F_n$ is a non-empty.

proof:-

let M be a complete Metric space and let sequence of (F_n) be a sequence of closed subset of M such that

$$F_1 \geq F_2 \geq F_3 \geq \cdots \geq F_n \geq \cdots \to (1)$$

and

$$d(F_n) \to 0 \quad \to (2)$$

part 1

Direct part:-

Assume that:-

M is complete

To prove that

$\bigcap_{n=1}^{\infty} F_n$ is non-empty.

(or) $\bigcap_{n=1}^{\infty} F_n$ is $\neq 0$.

for each +ve Integer n choose a point $x_n \in R$ $\to$ (2)

$x_n, x_{n+1}, x_{n+2}, \dots$ are lie in F_n

ie) $x_m \in F_{n_0}$ $\forall$ $m \geq n$ $\to$ (3)

Since $d(F_n) \to 0$

given $\varepsilon > 0$ there exists a +ve integer n_0 such that $d(F_n) < \varepsilon$ $\forall$ $n \geq n_0$

In particular $d(F_{n_0}) \to 0$ $\therefore d(F_{n_0}) < \varepsilon \to$ (4)

$\therefore d(x, y) < \varepsilon$ $\forall$ $x, y \in F_n$

Now $x_m \in F(n_0)$ $\forall$ $m \geq n_0$ [by 3]

$(m, n) \geq n_0$

$\Rightarrow (x_m, x_n) \in F_{n_0}$

$\Rightarrow d(x_m, x_n) < \varepsilon$ [by (4)]

$\therefore (x_n)$ is a cauchy sequence in M.

Since M is complete.

There exists a point $x \in M \ni : (x_n) \to x$

To prove $x \in \bigcap_{n=1}^{\infty} F_n$

Now, for any +ve integer n, $x_n, x_{n+1}, \dots$ is a sequence in F_n.

And this sequence convergence to x.

$x \in \overline{F_n}$ // by theorem 3.2

$x \in \bar{A}$ iff $(x_n) \to 0$

But F_n is a closed set and hence $F_n = \overline{F_n}$

$x \in F_n$

$x \in \bigcap_{n=\infty}^{\infty} F_n$

Hence $\bigcap_{n=1}^{\infty} F_n \neq \phi$

PART = II

Converse part:

Assume that :-

$$\bigcap_{n=1}^{\infty} F_n \neq \phi$$

To prove that.

M is complete.

Let (x_n) be any cauchy sequence in M.

Let $F_1 = \{x_1, x_2 \ldots x_n \ldots\}$

$F_2 = \{x_2, x_3 \ldots x_n \ldots\}$

$F_3 = \{x_3, x_4 \ldots x_n \ldots\}$

$\vdots$

$F_n = \{x_n, x_{n+1} \ldots\}$

Clearly,

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots \supseteq F_n \supseteq$

$\therefore \overline{F_1} \supseteq \overline{F_2} \supseteq \overline{F_3} \supseteq \ldots \supseteq \overline{F_n} \supseteq \ldots$

(F_n) is a decresing sequence of closed set.

(x_n) is a cauchy sequence.

Given $\epsilon > 0$ there exist a +ve integer n_0 such that

$d(x_n, x_m) < \epsilon \ \forall \ m, n \geq n_0$ for any Integer $n \geq n_0$

The distance between any two points of F_n is $< \epsilon$

$d(F_n) < \epsilon \ \forall \ n \geq n_0$

But $d(F_n) = d(\overline{F_n})$

$\therefore d(\overline{F_n}) < \epsilon \ \forall \ n \geq n_0 \rightarrow (5)$

$\therefore d(F_n) \rightarrow 0$

Hence $\bigcap_{n=1}^{\infty} F_n \neq \phi$

Let $x \in \bigcap_{n=1}^{\infty} (F_n)$

Then x and $x_n \in (\overline{F_n})$

$d(x_n, x) < \epsilon \ \forall \ n \geq n_0$

$(x_n) \rightarrow x$.

Hence M. is complete

Hence the cantor's intersection theorem.

Note: 1

In the above theorem $\bigcap_{n=1}^{\infty} F_n$ contains exactly only one points.

Suppose $\bigcap_{n=1}^{\infty} F_n$ contains two distinct points are x and y then.

$d(F_n) \geq d(x,y) \; \forall n$

$d(F_n)$ does not $\to 0$

$(\Rightarrow \Leftarrow)$

$\bigcap_{n=1}^{\infty} F_n$ contains exactly only one points.

Notes: 2.

In the above theorem $\bigcap_{n=1}^{\infty} F_n$ may be ϕ

If each (F_n) is not closed.

$\therefore F_n = (0, 1/n)$ in R.

clearly,

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots \dots \supseteq F_n \supseteq \dots \dots$

and $d(F_n) = 1/n \to 0$

and $n \to \infty$

But $\bigcap_{n=1}^{\infty}$ if ϕ

Note: 3

If the above theorem $\bigcap_{n=1}^{\infty} F_n$ may be ϕ. If the hypothesis $d(F_n) \to 0$

For example.

consider $F_n \neq [n, \infty)$ in R.

clearly.

F_n is a sequence of a closed set

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots \dots \supseteq F_n \supseteq \dots \dots$

also $\bigcap_{n=1}^{\infty} F_n \neq \phi$

Here $d(F_n) = \infty \; \forall n$ and hence hypothesis $d(F_n) \to 0$

Ex! 1

In R with usual metric $A = \{1, 1/2, \ldots 1/n\}$ is a no where dense.

proof:-

$\bar{A} = A \cup D(A)$

$= \{1, 1/2, 1/3, \ldots 1/n\} \cup \{0\}$

$\text{int } \bar{A} = \phi$

Ex! 2.

In any metric space M any non-empty subset of is not no where dense.

proof:-

In a discrete metric space every subset is both closed and.

$\bar{A} = \text{int } \bar{A}$

$= \text{int } A$

$= A$

$\therefore$ The ~~xxx~~ $\text{int } \bar{A} \neq \phi$

A is not no where dense

Ex! 3.

R with usual metric any finite set of A is a no where dense.

proof:-

Let A be a finite subset of R.

Then A is closed

and hence $A = \bar{A}$

Also since A is finite

No points of A is fiixal an int A.

$\text{int } \bar{A} = \text{int } A = \phi$

A is no where dense.

Ex:-

consider R with usual metric any $\{x\}$ is a no where dense.

proof:-

Any countable set of R.

Being a countable union of F. is

First category.

Note: 1

In particular Q is a first category.

Note: 2.

If A and B are set of First category in a metric space M. Then $A \cup B$ is also a First category.

proof:-

Since A and B are set of first category in a M ~~metric space X. Then A∪B~~ is ~~also a first category~~. We have $A = \bigcup_{n=1}^{\infty} E_n$ and $B = \bigcup_{n=1}^{\infty} H_n$

where E_n and H_n are no where dense subset in M.

$A \cap B$ is a countable union of no where dense subset of A.

$A \cup B$ is a first category.

Theorem: 3.6

Equivalent characterization for no where dense (or) Let M be a metric space and $A \subseteq M$. Then the following are equivalent.

i) A is nowhere dence in M.

ii) A is does not containing any non-empty open set.

iii) each non-empty open set has a non-empty open set disjoints for $\bar{A}$

proof:-

To excreise the radder.

Theorem: 3.7

Baire's category theorem:-

Statement:-

Any complete Metric space is of 2^{nd} category.

proof:-

Let M be a complete Metric space.

we claim that.

M is not of 1^{st} category

Let (A_n) be a sequence of nowhere dense set in M

To prove that:-

$$x \notin \bigcup_{n=1}^{\infty} A_n \neq M$$

Since M is open and A_1 is a no where dense.

There exist an open ball B_1 of radius less than 1.

such that B_1 disjoints from A_1

let F_1 denote the concentrate. closed bar closed radius $1/2$ times that of B_1

Now, int F_1 is open and A_2 is no where dense.

int F_2 contains an open ball B_2 radius $< 1/4$

such that B_2 disjoint from A_2

let F_3 be the concentric closed ball radius $1/2$ times of B_2.

Proceeding like this

we get the non-empty sequence closed ball

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

$$d(F_n) \to 0$$

Hence $d(M) \to 0$

as $n \to \infty$. since M is complete.

There exist the point $x \in M$. such that $x \in \bigcap_{n=1}^{\infty} F_n$

Also each $x \in F_n$ is disjoint from A_n hence $x \notin A_n \; \forall n$.

$$x \notin A_n \; \forall n$$

Hence $x \notin \bigcup_{n=1}^{\infty} A_n$

$$\bigcup_{n=1}^{\infty} A_n = M.$$

Hence M is of 2^{nd} category.

Hence the theorem.

Corollory:-

R is of 2^{nd} category.

proof:-

WKT R is complete Metric space

Hence R is 2^{nd} category

The converse of the above theorem isnot true

Any Metric space which is of 2^{nd} category need not be complete.

For ex:-

$M = R - Q$ the space of irrational number

WKT, Q is of 2^{nd} category.

ie) To prove that

$\emptyset$ is of 1^{st} category.

suppose M is of 2^{nd} category. Then $M \cup Q = R$ is also 1^{st} category.

Which is contradiction.

$\therefore$ M is of 2^{nd} category.

M. is not a closed subspace of R.

and hence M is not complete.

Pbm: 1

prove that any non-empty subset (a,b) in R is of 2^{nd} category.

Soln:-

L (a,b) be a non-empty in R.

Now $[a,b] = [a,b] \cup \{a\} \cup \{b\}$

$[a,b]$ is of 1^{st} category.

But $[a,b]$ is a complete Metric space

Hence is of 2nd category.

which is condradiction

2. prove that A closed set A in a Metric space M is nowhere dense Iff $\overline{A}$ is every where dense.

soln:-

let a be a closed set in M.

$\therefore A = \overline{A} \rightarrow (1)$

suppose A is no where dense in M.

$\therefore \text{Int } \overline{A} \neq \phi$

$\text{Int } A = \phi \quad (\text{by } (1))$

$\rightarrow (2)$

Now we claim that.

$\overline{A^c} = M.$

Obviously $\overline{A^c}$ subset of $M \rightarrow (3)$

Now let $x \in M$

let G be any open set such that $x \in G$.

Since $\text{int. } A = \phi$

we have $G \not\subset A$

$\therefore G \cap A^c \neq \phi$

$x \in \overline{A^c}$ (refer the corollary 3.2)

$M \subseteq \overline{A^c}$

by equ (2) & (3)

we have
$$M = \overline{A^c}$$
$\therefore$ Ac is every where dense in M.

conversely Ac be every where dense in M.

$\therefore \overline{Ac} = M$

we claim that int $A = \phi$

Let G be any non-empty open set in M.

since A^c is open.

we have $G \cap A^c \neq \phi$

$G \not\subset A$

They only open set which is contained in A is the non-empty set.

100

UNIT - III

Continuity:-

Limit:-

Let (M_1, d_1) and (M_2, d_2) be a Metric Space let $f :- M_1 \to M_2$ be a function. Let $a \in M$ and $l \in M_2$. The function f is said to be have a limit as $x \to a$ if given $\varepsilon > 0$ there exist $\delta > 0$ such that.

$$0 < d_1(x, a) < \delta$$

$$\Rightarrow d_2(f(x)\, l) < \varepsilon$$

we write.

$$\lim_{x \to a} f(x) . l$$

continuous:-

Let $(M_1 . d_1)$ and $(M_2 ; d_2)$ be a two Metric space. Let $a \in M_1$ a function $f : M_1 \to M_2$ to be continuous at a. If g_n $\varepsilon > 0$ there exict such that $d(x, a) < \delta$.

$$\Rightarrow d_2(f(x) . f(a) < \varepsilon$$

f is said to be continuous. If f is continuous at every point of M.

Note: 1

f is continuous at a iff $\lim_{x \to a}$ $f(x) = f(a)$

Note: 2.

The condition $d_1(x, a) < \delta$

$\Rightarrow d_2(f(x)\, f(a)) < \varepsilon$ can be written as

(i) $x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \delta)$

ii) $f(B(a, \delta)) \subseteq B(f(a), \varepsilon)$

Ex:
let $f: M_1 \to M_2$ be gn by $f(x) = a \in M_2$ is a fixed element Let $x \in M_1$ and $\varepsilon > 0$ be gn then for any $\delta > 0$ $f(B(x, \delta)) = \{a\} \subseteq \{a, \varepsilon\}$

$\therefore$ f is continuous at x we know $x \in M_1$ is arbitary.

f is continuous.

Theorem: 4.1
Let $(M_1 d_1)$ and $(M_2 d_2)$ be two Metric space. Let $(a \in M)$ a function $f: M_1 \to M_2$ is continuous at a Iff sequence $(x_n) \to a$ $\Rightarrow f(x_n) \to a$.

proof:
Assum that:-
suppose f is continuous at a.

Let (x_n) be a sequence is M_1. such that $(x_n) \to a$

we claim that.

$f(x_n) \to f(a) \to ①$

let $\varepsilon > 0$ be gn by defn. of continuity there exists $\delta > 0 \ni d(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon \to$ [From ①]

since $(x_n) \to a$

There exists a +ve Integer n_0 such that $d_1(x_n, a) < \varepsilon$ δ $\forall n \geq n_0$

Hence by equ ①

$(f(x_n)) \to f(a)$

conversely part:-
f is continuous at a

suppose f is not continuous at a.

Then there exists an $\varepsilon > 0 \ni : \forall \delta(\varepsilon) > 0$

$f(B(a, \delta)) \not\subset (f(a), \varepsilon)$

$d_1(x_1, a) < 1/n$

Choose $x_n \ni : x_n \in B(a, 1/n)$ particular and

$f(x_n) \in (f(a), \varepsilon)$

$d_1(x, a) < 1/n$ and

$d_2(f(x_n), f(a)) > \varepsilon$

$(x_n) \to a$ and $(f(x_n))$ is not convergent to $f(a)$

which is contradiction $\rightarrow\leftarrow$

f is continuous at a.

Corollary:

A function $f: M_1 \to M_2$ is continuous iff $(x_n) \to x_1 \Rightarrow (f(x_n)) \to f(x)$

We now characterize continuous map in terms open set

Theorem: 4.2.

Let (M_1, d_1) and (M_2, d_2) be two metric space $f: M_1 \to M_2$ is continuous iff $f^{-1}(G)$ is open M_1 whenever G is open in M_2 (or)

f is continuous, iff inverse image of every open set is open.

Proof:

Part I

Suppose f is continuous

Let G be a open set in M_2.

We claim that.

$f^{-1}(G)$ is open in M.

If $f^{-1}(G)$ is empty

Then it is open

Let $f^{-1}(G) \neq \phi$

let $x \in f^{-1}(G)$

hence $f(x) \in G$

Since G is open

There exists an open ball $\ni$.

$B(f(x), \varepsilon)$ such that $\subseteq G \rightarrow (1)$

Now. by defn. of continuity there exists an open ball $f(B(x, \varepsilon))$

$f(B(x, \delta)) \leq B(f(x), \varepsilon)$

$f(B(x, \delta_1) \subseteq$ or by (1))

$\therefore B(x, \delta) \subseteq f^{-1}(G)$

since $x \in f^{-1}(G)$ is arbitrary $f^{-1}(G)$ is open

conversly.

Suppose $f^{-1}(G)$ is open in M_2 whenever G is open in M_2

We ~~tal~~ claim that

f is continuous

let $x \in M$.

Now. $B(x, \varepsilon)$ is an open set in M_2

$f^{-1}(B(f(x)), \varepsilon)$ is open in M and

$x \in f^{-1}(B(f(x)\ \varepsilon)$

There exists $\delta > 0 \ni$.

$B(x . \delta) \subseteq f^{-1}(f(x); \varepsilon)$

f is continuous in M.

Since $x \in M_1$ is arbitary. f is continuous

Theorem: 4.3.

Let (M_1, d_1) and (M_2, d_2) be two Metric space fun/ $f: M_1 \to M_2$ is continuous iff $f^{-1}(F)$ is closed in M_1. whenever f is closed in M_2.

proof:-

Suppose $f: M_1 \to M_2$ is continuous

let $F \subseteq M_2$ we closed in M_2

f^c is open in M.

But $f^{-1}(f^c) = (f^{-1}(f))^c$

$\therefore f^{-1}(F)$ is closed in M.

conversely.

Suppose $f^{1}(F)$ is closed in M_1, whenever $f^{-1}(F)$ is closed in M_2

we claim that

f is continuous.

Let G be a open set in M_2.

$\therefore G^c$ is closed in M_2.

$\therefore f^{-1}(G^c)$ is closed in $M_{2,1}$

$\therefore (f^{-1}(G))$. G is closed in M_1

$\therefore f^{-1}(G)$ is open in M_1

f is continuous

we give one more characterization fun/. terms of closed set.

Theorem: 4.4

Let (M_1, d_1) and (M_2, d_2) be two Metric space then $f: M_1 \to M_2$ is continuous iff $f(A^c) \subset (f(A))^c \forall A \subseteq M_1$

proof:-

Suppose f is $A \subseteq M$.

Then $f(A) \subseteq M_2$

Since f is continuous.

$f^{-1}(f(n))$ is closed in the

Also $f^{-1} f(A) \supseteq A \; (f(A))^c \supseteq f(A)$

But A^c is the smallest closed set $\subseteq A$.

$\therefore \bar{A} \subseteq f^{-1}(f(A)^c)$

$\therefore f(A) \subseteq [f^{-1}(f(A))]^c$

$\therefore f(A^c) \subset [f(A)]$

conversly.

let $f(A^c) \subseteq (f(A))^c \; \forall A^c$

prove that

f is continuous we shall prove that

if f is closed set in M_2. Then $f^{-1} f$ is

closed in M.

By hypothesis

$[f(f^{-1}(F)]^c \subseteq (f f^{-1}(F))^c$

$\subseteq f$

$= F$ (since F is closed).

Thus $[f(f^{-1}(F))]^c \subseteq F$

$(f(F))^c \subseteq f^{-1}(F)$

also $f^{-1}(F) \subseteq f(x)$

$f^{-1}(F) = f^{-1}(F)^c$

Hence $f(F)$ is closed.

$\therefore f$ is continuous.

problem: 1

Let f be a continuous defined on a Metric Space n let $A = \{x \in M / f(x) \geq 0\}$ PT f is closed.

soln:-

$A = \{x \in M / f(x) \geq 0\}$

$= \{x \in M / f(x) \in [0, \infty)\}$

$= f^{-1} [0, \infty)$

also $[0, \infty)$ is a closed subset of R.

since f is continuous

$f^{-1} [0, \infty)$ is closed in M. A is closed

2. Show that the fun/ $f: R \to R$ defined by $f(x) = \{0$, if $x\}$ is irrational is not cotinuous by each of then following Method.

(i) By the causual ε, δ Method.

ii) By explitting a sequence (x_n) such that $(x_n) \to x$ and $f(x_n)$ no converges to $f(x)$

iii) By expliting and open set $G \ni f^{-1}(G)$ is not open

iv) By expliting a closed subset of V. $f^{-1}(G)$ is not closed

v) By expliting subset of A. $f(A^c)$ doesnot containing $+ . f(x)$

soln:- (i) To prove that

f is not contains at x

we have to there exists an $\varepsilon > 0 \ni$.

$\delta > 0$, $f(B(x)\delta) \not\subset B(x, \delta)$

5. Let (f, g) be a continuous real number value funⁿ on a metric space M. Let $A = \{x / x \in M$ and $f(x) < g(x)\}$ P.T A is open.

soln:-

Since f and G is of continuous real value fn on M.

And f, g is also continuous real value fn on M. Now,

$$A = \{x / x \in M \text{ and } f(x) < g(x)\}$$

$$A = \{x / x \in M \text{ and } f(x) - g(x) < 0\}$$

$$A = \{x / x \in M \text{ and } (f-g)x < 0\}$$

$$A = \{x / x \in M \text{ and } (f+g)x \in (\infty, 0)\}$$

$$= (f-g)$$

$(-\infty, 0)$ is open in R and $(f-g)$ is open in M.

$\therefore A$ is open in M.

6. If $f: R \to R$ and $g: R \to R$ are both continuous on R and if h R^2 is defined by $h(x, y) = f(x) - g(y)$. P.T h is cotinuous in R^2

soln:-

Let (x_n, y_n), (x_n, y_n) be a sequence in R^2 converging to (x, y)

$$(x_n, y_n) \to (x, y)$$

We prove that

$$(h(x_n, y_n)) \to h(x, y)$$

Since $(x_n, y_n) \to (x, y)$.

In R^2 $(x_n) \to x$ and $(y_n) \to y$ n R.

Also f and g are continuous

$$f(x_n) \to x.$$

$g(y_n) \to g(y)$

$(f(x_n), g(y_n)) \to f(x), g(x)$

$h(x_n, y_n) \to h(x, y)$

h is continuous on R^2

7\. Let (M, d) be a Metric space. Let $a \in M$ show that the func. $f: M \to R$ defined $f(x) = d(M)$ is continuous.

soln:-

Let $x \in M$

Let (x_n) be a sequence in $M \ni (x_n) \to x$

To prove that.

$f(x_n) \to f(x)$

Let $\varepsilon > 0$ be given

Now $|f(x_n) - f(x)| = |d(x_n, a) - d(x, a)|$

$\leq d(x_n, x)$

Since $(x_n) \to x$ there exists a +ve Integer n such that.

$d(x_n, x) < \varepsilon \ \forall n \geq n$

$f(x_n) \to f(x)$

f is continuous.

8\. Let f be a func from $R^2 \cdot R$ defined by $f(x, y) = x + (x_n)$ show that f is continuous in R^2.

proof:-

Let $(x, y) \in R^2$

Let (x_n, y_n) be a sequence in convergent to (x, y)

Then $(x_n) \to x$

$(y_n) \to y$

$f(x_n, y_n) = (x_n) \to x$

and $f(x_n, y_n) = (y_n) \to y$

$= f(x, y)$

$f(x_n, y_n) \to f(x, y)$

f is continuous.

defined $f: l_2 \to l_3$ as follows if x is l_2 is sequence $s_1, s_2 \ldots$ as $f(x)$ to the sequence $(0, s_1, s_2 \ldots s_T)$ f is continuous on l_2.

Soln:-

Let $y = (y_1, y_2 \ldots y_n \ldots) \subset l_2$

Let (x_n) be a sequence in l_3 converging to y.

Let $(x_n) = (x_{n1}, x_{n2}, \ldots x_{nx} \ldots)$

Then $(x_{n1}) \to y$

$(x_{n2}) \to y_2 \ldots (x_{nx}) \to y_x$

$f(x_n) = [0, x_{n1}, x_{n2}, \ldots x_n \ldots]$

$\to [0, y_1, y_2 \ldots y_k \ldots]$

$= f(y)$

$\therefore f(x_n) \to f(y)$

f is continuous.

10. Let G be a open subset of R. Prove that the characteristic fun. of G defined by

$$\varphi_G(x) = \begin{cases} 1 & \text{if } x \in G \\ 0 & \text{if } x \notin G \end{cases}$$ is continuous.

at every point of G.

Soln:-

Let $x \in G$

so that $\varphi_G(x) = 1$

Let $\varepsilon > 0$ be given

since G is open and $x \in G$ we can find a $\delta > 0 \ni$

$$B(x, \delta) \subseteq G$$

$$\varphi B(x, \delta) \subseteq \varphi_G(G)$$

$$= \{1\}$$

$$\subseteq B(1, \varepsilon)$$

$$\varphi_G(B(x, \delta)) \subset B(\varphi_G(G), \varepsilon)$$

φ_G is continuous at x

since $x \in G$ is arbitrary

φ_G is continuous on G.

Problem: 2

P.T the fun $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = 1/x$ is not uniformly continuous.

Soln:-

Let $\varepsilon > 0$ be given

Suppose there exists $\delta > 0 \ni : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Take $x = y + 1/2\, \delta$

Clearly $|x - y| = 1/2\, \delta < \delta$

$$= |f(x) - f(y)| < \varepsilon$$

$$= \left| 1/x - 1/y \right| < \varepsilon$$

$$= \left| \frac{1}{y + 1/2\,\delta} - 1/y \right| < \varepsilon$$

$$= \left| \frac{1}{2(y + 1/2)\delta y} \right| < \varepsilon$$

$$= \frac{\delta}{(2y + \delta)\, y} < \varepsilon$$

This inequality cannot be true for all $y \in (0,1)$

since $\frac{\delta}{(2y+\delta)y}$ becomes arbitary largest

y approches 0.

It is not uniformaly continuous.

3\. P.T the funct $f: R \to R$ defined by $f(x) = \sin x$ is uniformly continuous in R.

Soln:-

let $x, y \in R$ and $x < 0$

$\sin x - \sin y = |x-y| \cos z$

where $x > z > y$

$|\sin x - \sin y| = |(x-y)| \, |\cos z|$

$\leq |x-y| \quad (\because \cos z \leq 1)$

Hence for gn $\varepsilon > 0$ if we chose $\delta = \varepsilon$

we have $|x-y| < \delta$.

$\Rightarrow |f(x) - f(y)|$

$\Rightarrow |\sin x - \sin y| < \varepsilon$

$f(x)$ $\sin x$ is uniformly continuous in R.

Hence the problem.

U

UNIT - IV

connectedness:-

connected

Let (M, d) be a Metric space M is said to be connected if M connecte be representated as the union of two disjoint non-empty open sets

Disconnected:

If M is not connected is said to be disconnected.

Ex: 1

Let $M = [1,2] \cup [3,4]$ with usual metric then M is disconnected.

proof:-

$[1,2]$ and $[3,4]$ are open in M.

$M = A \cup B$

$M = [1,2] \cup [3,4]$

$\therefore [1,2] \neq \phi \; [3,4] \neq \phi$

$\therefore [1,2] \cap [3,4] \cap \phi$

M is union of two disjoint non empty open set namely $[1,2]$ $[3,4]$

M is connected.

Ex: 2.

Any discrete metric space M with more then one point is disconnected.

proof:-

Let A be a propen non-empty subset of M.

Since M has more than one point $\exists$ a set exists the A^c is also non-empty since M is discrete every subset of M is open.

A and A^c are open.

Thus $M = A \cup A^c$ where

A and A^c are two disjoint non-empty open set

$\therefore$ M is not connected.

Theorem : 5.1

Let (M, d) be a metric space then the following are equivalents.

(i) M is connected

ii) M is cannot to return us the union of two disjoint of the non-empty closed set.

iii) M cannot be return us the union of two non-empty set A and B $\ni$: $A \cap \bar{B} = \bar{A} \cap B = \phi$

iv) M is ϕ are the only set which are both open and closed in M.

proof:-

(i) $\Leftrightarrow$ (iii) suppose M is connected

Suppose (ii) is not connected

$M = A \cup B$ where A and B are closed

$A \neq \phi$ and $B \neq \phi$

$A \cap B \neq \phi$

$A^c = \phi$, $B^c = A$

B and A are open

B^c and A^c are open

B and A are open.

Thus M is the union of two disjoint non-empty open set.

M is not connected

which is contradiction. M is connected

(ii) $\Rightarrow$ iii)

Suppose iii) is not true. Then $M = A \cup B$ where $A \neq \phi$, $B \neq \phi$ and $A \cup \bar{B} = \bar{A} \cup B = \phi$.

We prove that A, and B are closed.

Let $x \in \bar{A}$

$x \notin B$ $(\bar{A} \cap B = \phi)$

$x \in A$ $(A \cup B = M)$

$\bar{A} \subseteq A$

$A = \bar{A}$ and hence A is closed

$|||^{ly}$ B is closed

Now $A \cap \bar{B} = \bar{A} \cap B$ $(A = \bar{A})$

$= \phi$

Thus $M = A \cup B$ where $A \neq \phi$, $B \neq \phi$

A and B are closed and $A \cup B = \phi$

which is contradiction to (ii)

iii) $\Rightarrow$ iv)

Suppose iv) is not open and closed. ~~and~~ ~~$\phi \neq A \neq \phi$~~ Then there exists $A \subseteq M \ni$ $A \neq M$ and $A \neq \phi$ and A is both open and closed.

Let $B = A^c$

Then B is also both open and closed and $B \neq \phi$

Also $M = A \cup B$

further $A \cap B = A \cap A^c$ (since $A = A$ and $B = A^c$)

$= \phi$

$|||^{ly}$ $A \cap B = \phi$

$M = A \cup B$ where $A \cap \bar{B} = \phi = \bar{A} \cap B$

which is condradiction to (ii)

(iii) $\Leftrightarrow$ iv)

iv) ⇒ ii)

Suppose M is not connected

$M = A \cup B$ where $A \neq \phi$, $B \neq 0$ A and B are open and $A \cap B = \phi$

Then $B^c = A$

Now, since B is open

∴ A is closed.

Also $A \neq 0$ and $A = M$ (since $B \neq \phi$)

A is not proper non-empty subset of M. which both open and closed.

⇒⇐

iv) ⇒ i)

The following theorem of given an equivalent characterization for the connectedness

Theorem: 5.2

A Metric space M is connected iff there does not exists a continous fun. $f: M$ on the disconnected Metric space $\{0, 1\}$.

proof:-

Suppose there exists a continuous fun $f: M \to \{0\}$

Since $\{0, 1\}$ is discrete.

$\{0\}$ $\{1\}$ are open

$A = f^{-1}(\{0\})$ and

$B = f^{-1}(\{1\})$ are open in M.

Since f is onto.

Clearly $A \cap B = \phi$

$A \cup B = M$.

Thus $M = A \cup B$ wher A and B are disjoint non-empty set

M is connected

which is $\Rightarrow \Leftarrow$

Hence there exists a continuous fun/

onto a conversly suppose.

M is not connected.

which is $\Rightarrow \Leftarrow$

Hence there exists a continuous fun

onto a conversly suppose

M is not connected

Then there exists disjoint non-empty open set A and B in M.

Such that $M = A \cup B$

Now define $f : M \to \{0\}$

by $f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$

clearly f is onto

Also $f^{-1}(\phi) = \phi$

$f^{-1}(\{0\}) = A$

$f^{-1}(\{1\}) = B$ and.

Thus the inverse image of every open $\{0\}$ is open in M.

Hence f is continuous.

These there exists f a is continuous fun. which is contradiction.

Hence M is connected.

Note:-

The above them can be represented as follows.

M is connected

iff every continuous fun

$f : M \to \{0, 1\}$ is not onto.

prob: 1

Let M be a metric space. Let A be a be a connected subset of M. If B is a subset of M such that $A \subseteq B \subset \bar{A}$. Then B is connected. In particular $\bar{A}$ is closed.

proof:-

Suppose B is not connected then

$$B = B_1 \cup B_2$$

where $B_1 \neq \phi$, $B_2 \neq \phi$, $B_1 \cap B_2 = \phi$ and B_1 and B_2 are open in B.

Now since B_1 and B_2 are open set in B.

There exists a open set G_1 and G_2 in M.

Such that $B_1 = G_1 \cap B$ and $B_2 = G_2 \cap B$

$$B = B_1 \cup B_2$$
$$= (G_1 \cap B) \cup (G_2 \cap B)$$
$$= (G_1 \cap G_2) \cap B$$
$$B \subseteq G_1 \cup G_2$$
$$A \subseteq (G_1 \cup G_2) \cap A$$
$$= (G_1 \cap A) \cup (G_2 \cap A)$$

Now, $G_1 \cap A$ and $G_2 \cap A$ are open in A.

further $(G_1 \cap A) \cap (G_2 \cap A)$
$$= (G_1 \cap G_2) \cap A.$$
$$= (G_1 \cap G_2) \cap B \quad (\text{since } A \subseteq B$$
$$= (G_1 \cap B) \cap (G_2 \cap B)$$
$$= B_1 \cap B_2$$
$$= \phi$$
$$(G_1 \cap A) \cap (G_2 \cap B) = \phi$$

since without less of generality let we assume that

$$G_1 \cap A = \phi$$

G_1 is open in M

We have

$G_1 \cap \bar{A} = \phi$

Since $G_1 \cap B = \phi$ (since $B \subseteq A$)

$B_1 = \phi$

$\Leftarrow \quad \Leftarrow$

B is connected.

2. If A and B are connected subset of a Metric space M and if $A \cap B \neq \phi$ prove that $A \cup B$ is connected

Let $f: A \cup B \to \{0,1\}$ be a continuous function.

Since $A \cap B \neq \phi$

we can choose

$x_0 \in A \cap B$

Let $f(x_0) = 0$

Since $f: A \cup B \to \{0,1\}$ is continuous

$f/A : A \to \{0,1\}$ is also continuous. But A is connected

Here $f(A)$ is not onto [by the 5.8]

$f(x) = 0 \ \forall \ x \in A$.

(or)

$f(x) = 1 \ \forall \ x \in A$.

But $f(x_0) = 0 \ \forall \ x \in A$.

$|||^{ly}$ $f(x) = 0 \quad \forall \ x \in B$

$f(x) = 0 \ \forall \ x \in A \cup B$.

This any continuous fun.

$f: A \cup B \to \{0,1\}$ is not onto

$A \cup B$ is connected.

Theorem: 5.2.

A subspace R is connected iff it is an interval.

proof:-

Let A be a connected subset of R.

suppose A is not an Interval.

Then there exists $a, b, c \in R \ni$.

$a < b < c$ and $a \in A$ but $c \notin A$.

Let $A_1 = (a, b) \cap A$ and

$A_2 = (-\infty, b) \cap A$

since $(-\infty, b)$ and (b, ∞) are open in R_1

A_1 and A_2 are open sets in A.

Also $A_1 \cap A_2 = \phi$ and

$A_1 \cup A_2 = A$

Futher $a \in A_1$ and $c \in A_2$.

Hence $A_1 \neq \phi$ and $A_2 \neq \phi$

Thus A is ~~union~~ union of two disjoint non-empty

open set A_1 and A_2.

Hence A is not connected

which is contradiction.

Hence A is an Interval.

conversely

A be an interval

we claim that

A is connected

suppose A is not connected

$A = A_1 \cup A_2$ where $A_1 \neq \phi$

$A_2 \neq \phi$

and A_1 and A_2 are closed sets in A

choose $x \in A_1$ and $x \in A_2$

since $A_1 \cap A_2 = \phi$

we have $x \neq z$

without less of generality

we assume that

$x < z$.

Now, since A is an Interval we have

$[x, z] \subseteq A$

$[x, z) \subseteq A_1 \cup A_2$

Every element of $[x, z]$ is either in A_1 (or) A_2

Now let,

$lub\{[x, z] \cap A_1\}$

clearly, $x \leq y \leq z$

Hence $y \in A$

Let $\varepsilon > 0$ be given then the definite lub if there exists.

$t \in (x, z) \cap A_1$ such that.

$y - \varepsilon < f \leq y$.

$(y - \varepsilon, y + \varepsilon) \cap A_1$, such that $y - \varepsilon < t \leq y$

$y \in [x, z] \cap A$

$y \in [x, z] \cap A_1$ [since $[x, z] \cap A$ is closed in A]

$\therefore y \in A_1 \rightarrow (1)$

Again by the defn. of y, $y + \varepsilon \in A_2 \; \forall$ $\varepsilon > 0$ such that $y + \varepsilon < z$

$y \in A_2$

$y \in A_2$ (since A_2 is closed) $\rightarrow (2)$

$y \in A_1 \cap A_2$ (by (1) & (2))

$\Rightarrow \Leftarrow$

$A_1 \cap A_2 \neq \phi$

Hence A is connected.

Theorem: 5.4

R is connected.

$R = (-\infty, \infty)$ is an Interval

R is connected.

1. To show that a subspaces of a connected Metric space need not to be connected.

proof:-

WKT, R is connected

$A = [1,2] \cup [3.4]$ is a subspace of R.

which is connected.

2. prove (or) R dispertive it A and c are connected subset's of a metric space. Subset R.

proof:-

we disprove the standered by giving a counter example.

let $A = [1,2]$, $B = [1,2] \cup [3.4]$

$C = R$.

clearly $A \subset B \subset C$

Hence A and c are connected.

But B is not connected.

Let M_1 be a connected Metric space. Let M_2 be any metric space. Let $M_1 \cap M_2$ be a continuously fun%. Then $f(M_2)$ is a connected Subset of M_2.

(or)

Any continuous image of a connected set is connected.

proof:-

Let $f(M_1) = A$.

So that f is function from M_1 onto A.

we claim that.

A is connected

suppose A is no connected

Then there exists a robber non-empty subset of ~~B~~, of A.

which is both open and closed in M_1

$f^{-1}(B)$ is a proper non-empty subset of M_1

which is both open and closed in M_1

Hence M_1 is not connected

$\Rightarrow \Leftarrow$

Hence A is connected.

State and prove intermitiate value.

Theorem:.

statement:-

Let f be a real valued continuous function defined by a interval of R take the every value of between any two values if assumes.

proof:-

Let a, b, c and $f(a) \neq f(b)$ without Lost generality

we assume that $f(a) < f(b)$

Let c be a $f(a) < c$, $f < b$.

The interval I is a connected subset of R

~~Inf~~ $f(I)$ is Interval [by theorem. 5.5]

Also $f(a)$, $f(b) \in f(I)$.

Hence $[f(a) . f(b)] \subseteq f(I)$

$c \in f(I)$ [Since $f(a) < c < f(b)$]

$c = f(x)$ for some $x \in I$.

fun. R. Then the range of R uncountable.

Soln:-

WKT.

R is connected.

Since f is a continuous fun. the f(R) is a connected subset of R.

$\therefore$ f(R) is an Interval of R.

also since f is a non-constant fun. the interval.

Henie Borel theorem:-

Statement:

Any closed interval [a,b] is compact subset of R.

proof:

let $\{G_\alpha / \alpha \in I\}$ be a family of open set in R such that $\bigcup_{\alpha \in I} G_\alpha \supseteq [a,b]$

let $S = \{x / x \in [a,b]$ and $[a,x]$ can be cover by a finite number G_α's $\}$

clearly $a \in S$ and hence S ~~...~~ $x \neq \phi$

also S is bounded above by let c denote the upper bound of S.

clearly $c \in [a,b]$

$\therefore c \in G_{\alpha_1}$ for some $\alpha \in I$

since G_{α_1} is open there exist $\epsilon > 0$ $(c-\epsilon, c+\epsilon)$ G_{α_1} choose $x_1 \in [a,b]$ such that $x_1 < x$ and $[x_1, x] \subseteq G_\alpha$

Now since $x_1 < c_1$ $[a_1, x_1]$ can be coverd by a finite no of G_{α_1}. The finite number of G_{α_i}'s together with G_{α_1} covered $[a, c]$

The defn of S, $c \in S$.

Now we claim that

Such that $x_2 > c$ and $[c_1, x_2] \subseteq G_{\alpha_1}$ as before $[a_1, x_2]$ can be coveded by finite no of G_α's

Hence $x_2 \in S$. but $x_2 > c$

which